Approximate reconstruction from circular mean data via classical summability

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Notation:

\( x = (x_1, x_2) \in \mathbb{R}^2 \), also use \( y, \xi \) etc.

Scalar (inner, dot) product: \( \langle x, y \rangle = x_1 y_1 + x_2 y_2 \).

\( u = u(\theta) = (\cos \theta, \sin \theta) \),

\( v = u^\perp = u(\theta + \pi/2) \).
Radon transform:

\[ Rf(u, t) = \int_{-\infty}^{\infty} f(tu + sv) \, ds = f_u(t). \]

Applications are well known.
Circular mean transform:

\[ Mf(\xi, r) = \int_{0}^{2\pi} f(\xi + ru(\phi)) d\phi \]

Models the data acquisition scheme in photoacoustic, also known as thermoacoustic, tomography that is currently being tested for possible clinical applications.
Typical assumptions are that $f$ has support in the unit disk and $\xi = u(\theta)$, $0 \leq \theta < 2\pi$.

We are interested in reconstruction of $f$ in terms of the data $Mf(\xi, r)$, $\xi \in \Xi$, where $\Xi$ is some appropriate collection of detectors exterior to the unit disk. An important special case is $\Xi = S^1$ the unit circle, the boundary of the unit disk with data $Mf(u(\theta), r)$, $0 \leq \theta < 2\pi$.

In the case $\Xi = S^1$ exact inversion formulas are known but, as in the case of the classical Radon transform, need to be regularized for numerical work.
Inversion formulas for $f$ in terms of the data $\mathcal{M} f(u(\theta), r)$, $0 \leq \theta < 2\pi$, were first published in FHR = D. Finch, M. Haltmeier, and Rakesh, Inversion of spherical means and the wave equation in even dimensions, *SIAM J. Appl. Math.* 68, no. 2, (2007), 392-412.

An alternate derivation and further generalizations can be found in Y. A. Antipov, R, Estrada, and B. Rubin, Inversion formulas for spherical means in constant curvature spaces, (2011) preprint.


Recall the notion of a *ridge function*:

\[ H(x) = h(\langle x, u \rangle) \]

If \( f(x) \) has compact support

\[
H \ast f(x) = \int_{\mathbb{R}^2} h(y)f(x - y)dy \\
= \int_{-\infty}^{\infty} h(t)Rf(u, \langle x, u \rangle - t)dt \\
= h \ast f_u(\langle x, u \rangle).
\]
If \( K(x) \) is a sum of ridge functions, i. e.

\[
K(x) = \int_0^{2\pi} h(\langle x, u(\theta) \rangle) \frac{d\theta}{2\pi}
\]

then

\[
K \ast f(x) = \int_0^{2\pi} \left\{ \int_{-\infty}^{\infty} h(t) Rf(u(\theta), \langle x, u(\theta) \rangle - t)dt \right\} \frac{d\theta}{2\pi}
\]

(1)

\[
= \int_0^{2\pi} h \ast f_{u(\theta)}(\langle x, u(\theta) \rangle) \frac{d\theta}{2\pi}.
\]

If \( K \) is a an approximation of the identity then (1) gives rise to a reconstruction algorithm for \( f \) in terms of its Radon transform data.

Remark: There is a formula for \( h \) in terms of \( K \) that can be explicitly evaluated in certain cases.
Typical examples:

\begin{align*}
\text{(2)} & \quad K(x) = \frac{1}{\pi} \begin{cases} 
1 \quad \text{if } |x| \leq 1 \\
0 \quad \text{otherwise}
\end{cases} \\
& \quad h(t) = \frac{1}{\pi} \begin{cases} 
1 \quad \text{if } |t| \leq 1 \\
1 - |t|/(t^2 - 1)^{1/2} \quad \text{otherwise,}
\end{cases}
\end{align*}

\begin{align*}
\text{(3)} & \quad K(x) = \frac{3}{\pi} \begin{cases} 
1 - |x| \quad \text{if } |x| \leq 1 \\
0 \quad \text{otherwise}
\end{cases} \\
& \quad h(t) = \frac{3}{\pi} \begin{cases} 
1 - \frac{\pi}{2} |t| \quad \text{if } |t| \leq 1 \\
1 - t \arcsin(1/t) \quad \text{otherwise.}
\end{cases}
\end{align*}
\[ K(x) = \frac{1}{2\pi} \frac{1}{(1 + |x|^2)^{3/2}} \]
\[ h(t) = \frac{1}{2\pi} \frac{1 - t^2}{(1 + t^2)^2}, \]

Note that in all the above cases the family of functions parametrized by \( \epsilon \)

\[ K_\epsilon(x) = \frac{1}{\epsilon^2} K\left(\frac{x}{\epsilon}\right) \]

are well known approximations of the identity as \( \epsilon \to 0 \). The corresponding functions \( h_\epsilon(t) \) of course, are given by

\[ h_\epsilon(t) = \frac{1}{\epsilon^2} h\left(\frac{t}{\epsilon}\right) \]
We use the same philosophy to reconstruct $f$ from its circular mean transform data.

$G$ is radial with center $\xi$:

$$G(x) = g(|x - \xi|)$$

If $f$ has compact support

$$\int_{\mathbb{R}^n} G(x)f(x)dx = \int_0^\infty \int_0^{2\pi} g(r)f(\xi + ru(\theta))rd\theta dr$$

$$= \int_0^\infty g(r)Mf(\xi, r)rdr.$$
If $K(x, y)$ is a sum of radial functions in the variable $y$, i.e.

$$K(x, y) = \int_{\Xi} k(x, \xi, |y - \xi|) d\mu(\xi)$$

then

$$\int_{\mathbb{R}^2} K(x, y) f(y) dy = \int_{\Xi} \left\{ \int_0^\infty k(x, \xi, r) Mf(\xi, r) r dr \right\} d\mu(\xi).$$

If $f$ is sufficiently regular and has support in a region $\Omega$ and $K(x, y)$ is a good approximation of the identity in $y$ at each $x \in \Omega$ then identity (5) represents an approximate reconstruction of $f$ in terms of the data $Mf(\xi, r)$, $\xi \in \Xi$ and $r > 0$. 
Such a kernel $K(x, y)$ can be conveniently viewed as a family of functions in the $y$ variable parameterized by $x$, each member of which is a sum of radial functions with centers in $\Xi$.

As alluded to earlier, we will study the case $\Xi = S^1 = \{\xi = u(\theta) : 0 \leq \theta < 2\pi\}$ with $d\mu(\xi) = \frac{d\theta}{2\pi}$ and $f$ supported in $B = \{x : |x| < 1\}$.

Remark: I don’t know how to solve

$$G(y) = \int_{S^1} g(|y - u|)d\mu(u)$$

for $g$ and $\mu$ in terms of $G$.

This leaves us with the problem of how to construct such $K(x, y)$ and $k(x, \xi, |y - \xi|)$ pairs?
Note that
\[
\lim_{r \to \infty} \left\{ |x - ru| - |y - ru| \right\} = \langle y - x, u \rangle.
\]

This suggests that, roughly speaking, if the detector \( \xi = ru \) is relatively far from \( x \) and \( y \) then \( |x - \xi| - |y - \xi| \) looks like \( \langle y - x, u \rangle \).

We know that
\[
\epsilon^{-2} K\left((y - x)/\epsilon\right) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\left(\frac{\langle y - x, u(\theta) \rangle}{\epsilon}\right) \frac{d\theta}{2\pi}
\]
is a good approximation of the identity at \( x \) with an appropriate choice of \( h \). i.e. one of the examples of \( K, h \) pairs.

Hence, it is not unreasonable to expect that
\[
K_1(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\left(\frac{|x - u(\theta)| - |y - u(\theta)|}{\epsilon}\right) \frac{d\theta}{2\pi}
\]
where \( h \) comes from the ridge function representation of a kernel \( K \), i.e. one of the examples (3) or (4), looks like a summability kernel or approximate identity at \( x \). At least for \( x \) and \( y \) close to the origin.
Plots of $K_1(x, y; \epsilon)$ for fixed $x$ and $\epsilon$ as a function of $y$. Here $h(t) = \frac{1}{2\pi} \frac{1-t^2}{(1+t^2)^2}$ as in example (4).

$x = (0, 0), \epsilon = 0.5$.  

$x = (0, 0), \epsilon = 0.25$.  

$x = (0, 0), \epsilon = 0.125$.  

$x = (0, 0), \epsilon = 0.0625$. 
More plots of $K_1(x, y; \epsilon)$ for fixed $x$ and $\epsilon$ as a function of $y$ with the same $h(t)$.

$x = (0, 0), \ \epsilon = 0.0625.$

$x = -\frac{1}{2\sqrt{2}}(1, 1), \ \epsilon = 0.0625.$

$x = -\frac{3}{4\sqrt{2}}(1, 1), \ \epsilon = 0.0625.$

$x = -\frac{7}{8\sqrt{2}}(1, 1), \ \epsilon = 0.0625.$
Phantom and detectors.

We use a discretization of

$$
\int_{|y|<1} K_1(x, y; \epsilon)f(y)dy
= \int_0^{2\pi} \left\{ \int_0^2 h_\epsilon(|x - u(\theta)| - r)Mf(u(\theta), r)rdr \right\} \frac{d\theta}{2\pi}.
$$
Reconstruction

\[ \tilde{f}(x) = \frac{C}{MN} \sum_{j=1}^{N} \left\{ \sum_{i=1}^{M} h_{\epsilon}(|x - u_{\theta_j} - r_i|) \mathcal{M} f(u_{\theta_j}, r_i) r_i \right\} \]

\[ h_{\epsilon}(t) = \frac{\epsilon^2 - t^2}{(\epsilon^2 + t^2)^2}, \quad \epsilon = 0.01, \quad M = 299, \quad N = 300. \]

Data and reconstruction.
These and similar numerical experiments suggest that $K_1(x, y; \epsilon)$ is a summability kernel and a good approximation of the identity at $x$ for $|x| < 1$ as a function of $y$, $|y| < 1$, for sufficiently small $\epsilon$.

Further numerical experiments suggest that the set of detectors $\Xi$ need not be restricted to circles. For example

$$K(x, y; \epsilon) = C \sum_{\xi_i \in \Xi} h_\epsilon(|x - \xi_i| - |y + \xi_i|)$$

will still be a good approximation of the identity at $x$ for $|x| < 1$ as a function of $y$, $|y| < 1$, for appropriate $\epsilon$ as long as, roughly speaking, the set of detectors $\Xi$ is a sufficiently dense set surrounding the unit disk. F. Filbir, R. Hielscher, and W.R. Madych, Reconstruction from circular and spherical mean data, *Applied and Computational Harmonic Analysis* 29, (2010), 111-120.
Is it true that

$$\lim_{\epsilon \to 0} \int_{|y|<1} K_1(x, y; \epsilon) f(y) dy = f(x)$$

whenever $f$ is bounded, vanishes outside the unit disk, and is continuous at $x$?

To hopefully simplify the matter try working with

$$K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\left(\frac{|x - u(\theta)|^2 - |y - u(\theta)|^2}{2\epsilon}\right) \frac{d\theta}{2\pi}$$

which is also a sum of radial functions in the variable $y$, seems pretty much like $K_1(x, y; \epsilon)$, but the argument $|x - u(\theta)|^2 - |y - u(\theta)|^2$ is algebraically easier to work with.
Note that \( K_2(x, y; \epsilon) \) can be re-expressed as

\[
K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h \left( \frac{x - y}{\epsilon}, u(\theta) - \frac{x + y}{2} \right) \frac{d\theta}{2\pi}
\]

or

\[
K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h \left( \frac{x - y}{\epsilon}, u(\theta) \right) + \frac{|y|^2 - |x|^2}{2} \frac{d\theta}{2\pi}.
\]
Plots of $K_2(x, y; \epsilon)$ for fixed $x$ and $\epsilon$ as a function of $y$ with $h(t)$ as in (4).

$x = (0, 0), \epsilon = 0.0625.$

$x = -\frac{1}{2\sqrt{2}}(1, 1), \epsilon = 0.0625.$

$x = -\frac{3}{4}(1, 1), \epsilon = 0.0625.$

$x = -\frac{7}{8}(1, 1), \epsilon = 0.0625.$
Comparison of the plots of $K_1(x, y; \epsilon)$ and $K_2(x, y; \epsilon)$ for fixed $x$ and $\epsilon$ as a functions of $y$.

$K_1$ at $x = -\frac{7}{8\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.

$K_2$ at $x = -\frac{7}{8\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$. 
Theorem: If $h$ is the function in example (4), that is
\[ h(t) = \frac{1}{2\pi} \frac{1 - t^2}{(1 + t^2)^2}, \]
then
\[ \lim_{\epsilon \to 0} \int_{|y| < 1} |y| < 1 K_2(x, y; \epsilon)f(y)dy = c(x)f(x) \]
where
\[ c(x) = \frac{\pi}{1 - |x|^2} \]
whenever $f$ is bounded, vanishes outside the unit disk, and is continuous at $x$.

This is a corollary of the following:

Lemma: If $h$ is the function in example (4) then
\[ \left| \frac{1}{2\pi} \int_0^{2\pi} h(\langle z, u(\theta) - x\rangle) d\theta \right| \leq \frac{C}{1 + |z|^3} \]
for all $z \in \mathbb{R}^2$ where $C$ is a constant that depends only on $x$ when $|x| < 1$. 
Proof of Lemma:
Use residues to evaluate the integral and get

\[ \int_0^{2\pi} h(\langle z, u(\theta) - x \rangle) d\theta = c \text{Re} \frac{\langle z, x \rangle + i}{((\langle z, x \rangle + i)^2 - |z|^2)^{3/2}}. \]

Follow this by several pages of algebraic manipulations together with applications of appropriate inequalities to get the desired result.

Note 1: The analytic nature of \( h(t) \) and the change to \(|x - \xi|^2 - |y - \xi|^2\) allowed us to do this.

Note 2: Want argument which depends only on the integrability of

\[ K(x) = \int_0^{2\pi} h(\langle x, u(\theta) \rangle) d\theta \]

over \( \mathbb{R}^2 \).
Corollary 1 of Theorem: If $x$ is in the unit disk $B$, and $f$ is in $C^2(B)$ then

$$
\begin{align*}
(6) \quad \frac{f(x)}{(1-|x|^2)} &= -\frac{1}{\pi^2} \int_0^{2\pi} \left\{ \frac{1}{2} \int_0^{\sqrt{2}a} \left\{ M(r) + M(\sqrt{2a^2 - r^2}) - 2M(a) \right\} \frac{r}{(r^2 - a^2)^2} \, dr \\
&\quad + \int_0^{\infty} M(r) \frac{r}{(r^2 - a^2)^2} \, dr - \frac{M(a)}{a^2} \right\} d\theta
\end{align*}
$$

where $M(r) = Mf(\xi, r)$ and $a = |x - u(\theta)|$.

Remark: $f$ in $C^2(B)$ is overkill. I suspect that $f$ in $L^p(B)$ for $p \geq p_0$ is sufficient.
Corollary 2 of Theorem: If $\Omega$ is the unit disk, $x$ is in $\Omega$, and $f$ is in $C^2(\Omega)$ then $f(x) =$

$$\frac{1 - |x|^2}{2\pi} \int_0^{2\pi} \left\{ \int_0^2 \log(|r^2 - |x - u(\theta)|^2|) \frac{d}{dr} \left( \frac{1}{2r} \frac{dMf(u(\theta), r)}{dr} \right) \, dr \right\} \, d\theta.$$

Remark 1: Analogues of the Theorem and Corollaries are valid when the unit disk is replaced by a more general elliptical region and the set of detectors $\Xi$, currently a circle, is replaced by a more general ellipse. In this case $d\mu(\xi)$ is not the usual arclength.

Remark 2: The inversion formula of the Corollary should be compared with the inversion formula in FHR.

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^2 \log(|r^2 - |x - u(\theta)|^2|) \frac{d}{dr} \left( r \frac{dMf(u(\theta), r)}{dr} \right) \, dr \, d\theta.$$
Analogous results are valid for spherical mean transforms in higher dimensions $n$.

In fact, in the case $n = 3$ a stronger version of the Theorem is valid. The function $h$ need not be analytic. It suffices that

$$K(x) = \frac{1}{4\pi} \int_{S^2} h(\langle x, u \rangle) d\sigma(u)$$

be integrable over $\mathbb{R}^3$. 