The X-ray transform on constant curvature disks

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Toy Model: The X-ray transform on CCD’s

Let $M$ the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1, 1)$ define the metric $g_\kappa(z) := (1 + \kappa|z|^2)^{-2}|dz|^2$ on $M$, of constant curvature $4\kappa$. 

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\kappa = -0.8
\]

A family of **simple** metrics which degenerates at $\kappa \to \pm 1$. 
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Introduction

The XRT on media with variable refractive index

The general project is to understand the XRT on manifolds. Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc.

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible!).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).
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Parallel v/s fan-beam geometry

Parallel geometry: enjoys the Fourier Slice theorem, which allows for a rigorous, efficient regularization theory.

Fan-beam geometry:
- 'natural' acquisition geometry, then traditionally rebinned into parallel data before processed. [Natterer '01]
- no parallel geometry on non-homogeneous surfaces. Instead, PDE’s on the unit phase space.

The Euclidean disk benefits from both viewpoints.
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Literature, or rather, authors...

Radon, Funk, Helgason, Ludwig, Gel’fand, Graev, Quinto, Cormack, Natterer, Maass, Louis, Rigaud, Hahn, Kuchment, Agranovsky, Ambartsoumian, Krishnan, Abishek, Mishra
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THE RADON TRANSFORM ON A FAMILY OF CURVES IN THE PLANE¹

A. M. CORMACK

Abstract. Inversion formulas are given for Radon’s problem when the line integrals are evaluated along curves given, for a fixed \((p, \phi)\), by \(r^a \cos|\alpha(\theta - \phi)| = p^a\), where \(\alpha\) is real, \(\alpha \neq 0\).
1. The Euclidean case

2. Const. Curv. Disks: Range Characterization

The X-ray transform on constant curvature disks

The Euclidean case

The classical moment conditions

Parallel geometry: \( \mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R} \times \mathbb{S}^1) \)

\[
\mathcal{R} f(s, \theta) = \int_{\mathbb{R}} f(-s \hat{\theta}^\perp + t \hat{\theta}) \, dt, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^1.
\]

Moment conditions: Gelfand, Graev, Helgason, Ludwig

\( \mathcal{D}(s, \theta) = \mathcal{R} f(s, \theta) \) for some \( f \) iff

(i) \( \mathcal{D}(s, \theta) = \mathcal{D}(-s, \theta + \pi) \) for all \( (s, \theta) \in \mathbb{R} \times \mathbb{S}^1 \).

(ii) For \( k \geq 0 \),

\[
p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s, \theta) \, ds = \sum_{k=0}^k a_{k,k} e^{ik\theta}.
\]

\[
\Rightarrow \int_{\mathbb{R}} \int_{\mathbb{S}^1} \mathcal{D}(s, \theta) s^k e^{ik\theta} \, ds \, d\theta = 0, \quad |p| \geq k, \quad p - k \text{ even}.
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\[\equiv \int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{D}(s, \theta) s^k e^{ip\theta} \, ds \, d\theta = 0, \quad |p| > k, \; p - k \text{ even}.\]
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$D(s, \theta) = \mathcal{R}f(s, \theta)$ for some $f$ iff

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$\Leftrightarrow \int_{\mathbb{S}^1} \int_{\mathbb{R}} D(s, \theta)s^k e^{ip\theta} \, ds \, d\theta = 0, \ |p| > k, \ p - k \ \text{even.}$
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The X-ray transform on constant curvature disks

The Euclidean case

The Pestov-Uhlmann range characterization

\[ l_0 : C^\infty(M) \rightarrow C^\infty_+ (\partial_+ SM) \]

\[ l_0 f(x, v) = \int_0^{\tau(x, v)} f(\gamma_{x,v}(t)) \, dt. \]

\( S \): scattering relation

Range characterization of \( l_0 \):

\[ l_0(C^\infty(M)) = P_-(C^\infty_\alpha(\partial_+ SM)), \]

\([\text{Pestov-Uhlmann '05}]\)

\( P_- \) takes the form \( P_- := A^*_+ H_- A_+ \), where

\[ \bullet A_+ : C^\infty(\partial_+ SM) \rightarrow C^\infty(\partial SM) \text{ symmetrization w.r.t. } S. \]

\[ \bullet H_- : \text{odd Hilbert transform on the fibers of } \partial SM. \]

\[ \bullet A^*_+ : C^\infty(\partial SM) \rightarrow C^\infty(\partial SM): A^*_+ f(x, v) = f(x, v) - f(S(x, v)). \]
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\[ (M, g) \]

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The Euclidean case

Equivalence of ranges characterizations

**Theorem (M., IPI, ’15)**

*Both range characterizations above are equivalent.*

Sketch of proof: Understand the operator $P_- = A^* H A_+$.

- Euclidean scattering relation:
  \[ S(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha). \]
- Explicit construction of the SVD of $P_- : L^2(\theta + SM) \rightarrow L^2(\theta + SM)$.
- Reparameterized moment conditions is equivalent to saying “$D \perp \text{Range } P_-$.”

We also understand how the cokernel can be realized by other types of integrands.
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- Reparameterized moment conditions is equivalent to saying “$\mathcal{D} \perp \text{Range } P_-$”.

\[ u'_{p,q} = e^{ip\beta} (e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha}) \]

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**The Euclidean case**

**SVD of** $I_0$

**Zernike polynomials:**
$Z^{n,k}$, $n \in \mathbb{N}_0$, $0 \leq k \leq n$.

Uniquely defined through the properties:

[Kazantzev-Bukhgeym '07]

- $Z_{n,0} = z^n$.
- $\partial_z Z_{n,k} = -\partial_z Z_{n,k-1}$,
  $1 \leq k \leq n$.
- $Z_{n,k}|_{\partial M(e^{i\beta})} = e^{i(n-2k)\beta}$.

In addition,

$$\langle Z_{n,k}, Z_{n',k'} \rangle_{L^2(M)} = \frac{\pi}{n+1} \delta_{n,n'} \delta_{k,k'}.$$ 

$$I_0[Z^{n,k}] = \frac{C}{n+1} e^{i(n-2k)(\beta+\alpha+\pi)} (e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}).$$ 

(in parallel coordinates, $\beta + \alpha + \pi = \theta$ and $\sin \alpha = s$)
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$$(Z_{n,k}, Z_{n',k'})_{L^2(M)} = \frac{\pi}{n+1} \delta_{n,n'} \delta_{k,k'}.$$
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Outline

1. The Euclidean case

2. Const. Curv. Disks: Range Characterization

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Const. Curv. Disks: Range Characterization

Range characterizations: statement

Theorem (Mishra-M., preprint ’19)

Let $M$ equipped with the metric $g_{\kappa}$ for $\kappa \in (-1, 1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S^*_Au = u$. Then $u \in l_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ for some $w \in C^\infty_{\alpha}(\partial_+ SM)$, [Pe-Uhl, ’04]

2. $u$ satisfies a complete set of orthogonality/moment conditions:

   $$(u, \psi^{\kappa}_{nk})_{L^2(\partial_+ SM, d\Sigma)} = 0, \quad n \geq 0, \quad k < 0, \quad k > n,$$

where we have defined

$$\psi^{\kappa}_{nk} := \frac{(-1)^n}{4\pi} \sqrt{s^{\kappa}_n(\alpha)} e^{(n+2k)(\beta+s^{\kappa}_n(\alpha))} (e^{(n+1)s^{\kappa}_n(\alpha)} + (-1)^n e^{-(n+1)s^{\kappa}_n(\alpha)})$$

$$s^{\kappa}_n(\alpha) := \tan^{-1} \left( \frac{1 - \kappa}{1 + \kappa} \tan \alpha \right)$$

$$s^{\kappa}_0(\alpha) = \alpha, \quad s^{\kappa} \circ s_{-\kappa} = \text{id}$$

3. $C_- u = 0$, where $C_- := \frac{1}{4\pi} H^* A_{\kappa} - (\text{id} + C_2)^2 = \text{Proj}_{\text{ran} I_0}$
Range characterizations: statement

**Theorem (Mishra-M., preprint ’19)**

Let $M$ equipped with the metric $g_\kappa$ for $\kappa \in (-1, 1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in l_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ for some $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. [Pe-Uhl, ’04]

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\psi_{n,k}^\kappa := \frac{(-1)^n}{4\pi} \sqrt{s'_\kappa(\alpha)} e^{i(n-2k)(\beta + s_\kappa(\alpha))} (e^{i(n+1)s_\kappa(\alpha)} + (-1)^n e^{-i(n+1)s_\kappa(\alpha)}),
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s_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa} \tan \alpha\right). \quad (s_0(\alpha) = \alpha, \quad s_\kappa \circ s_{-\kappa} = id)
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s_\kappa(\alpha) := \tan^{-1}\left(\frac{1 - \kappa}{1 + \kappa} \tan \alpha\right). \quad (s_0(\alpha) = \alpha, \quad s_\kappa \circ s_{-\kappa} = id)
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3. $C_- u = 0$, where $C_- := \frac{1}{2} A^* H A_-$
Range characterizations: statement

Theorem (Mishra-M., preprint ’19)

Let $M$ equipped with the metric $g_{\kappa}$ for $\kappa \in (-1, 1)$. Suppose $u \in C^{\infty}(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in l_0(C^{\infty}(M))$ iff either of the following is satisfied

1. $u = P_- w$ for some $w \in C^{\infty}_{\alpha, +, -}(\partial_+ SM)$. [Pe-Uhl, ’04]
2. $u$ satisfies a complete set of orthogonality/moment conditions:
   \[(u, \psi_{n,k}^{\kappa})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \quad n \geq 0, \quad k < 0, \quad k > n,\]
   where we have defined
   \[\psi_{n,k}^{\kappa} := \frac{(-1)^n}{4\pi} \sqrt{s'_\kappa(\alpha)} e^{i(n-2k)(\beta+s_\kappa(\alpha))} (e^{i(n+1)s_\kappa(\alpha)} + (-1)^n e^{-i(n+1)s_\kappa(\alpha)}),\]
   \[s_\kappa(\alpha) := \tan^{-1} \left( \frac{1 - \kappa}{1 + \kappa \tan \alpha} \right). \quad (s_0(\alpha) = \alpha, \quad s_\kappa \circ s_{-\kappa} = id)\]
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$(id + C_-^2 = \Pi_{\text{Ran } l_0})$
Theorem (Mishra-M., preprint ’19)

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3. $C_-u = 0$, where $C_- := \frac{1}{2}A^*H_-A_-$ \quad (id + C_-^2 = \Pi_{\text{Ran} \ l_0})
Proof

In light of the first item, understand the action of $P_- = A^* H_- A_+$, e.g. find its SVD. Construct functions that

- extend smoothly under $A_\pm$
- transform well under fiberwise Hilbert transform and scattering relation

$$S(\beta, \alpha) = (\beta + \pi + 2\varsigma_\kappa(\alpha), \pi - \alpha).$$

- are even or odd w.r.t. $S_A := S \circ (\alpha \mapsto \alpha + \pi)$

This produces four families of functions, some giving the $L^2 - L^2$ SVD of $P_-$ and the eigendecomposition of $C_-$. In particular, $\text{Ran } P_- = \ker C_-$. The SVD picture is identical to the Euclidean one!
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Outline

1. The Euclidean case
2. Const. Curv. Disks: Range Characterization
The X-ray transform on constant curvature disks

Const. Curv. Disks: Singular Value Decomposition

SVD: Statement

**Theorem (Mishra-M., preprint ’19)**

Let $M$ be the unit disk equipped with the metric $g_\kappa$ for $\kappa \in (-1, 1)$. Define $s_\kappa(\alpha)$ and $\{\psi^\kappa_{n,k}\}_{n \geq 0, k \in \mathbb{Z}}$ as above, as well as

$$\hat{Z}_{n,k}^\kappa(z) := \sqrt{\frac{n+1}{\pi}} \left(1 - \kappa\right) \frac{1 + \kappa|z|^2}{1 - \kappa|z|^2} \frac{Z_{n,k}}{Zernike} \left(\frac{1 - \kappa}{1 - \kappa|z|^2} z\right),$$

$$\hat{\psi}_{n,k}^\kappa := 2\sqrt{1 + \kappa} \psi_{n,k}^\kappa, \quad \sigma_{n,k}^\kappa := \frac{1}{\sqrt{1 - \kappa}} \frac{2\sqrt{\pi}}{\sqrt{n+1}}.$$

- $\{\hat{Z}_{n,k}^\kappa\}_{n \geq 0, 0 \leq k \leq n}$ ONB of $L^2(M, w_\kappa)$ where $w_\kappa(z) := \frac{1 + \kappa|z|^2}{1 - \kappa|z|^2}$.
- $\{\hat{\psi}_{n,k}^\kappa\}_{n \geq 0, 0 \leq k \leq n}$ ON in $L^2(\partial^+ SM, d\Sigma^2) \cap \ker(id - S_A^*)$.

For any $f \in w_\kappa L^2(M, w_\kappa)$ expanding as

$$f = w_\kappa \sum_{n \geq 0} \sum_{k=0}^{n} f_{n,k} \hat{Z}_{n,k}^\kappa,$$

we have

$$l_0 f = \sum_{n \geq 0} \sum_{k=0}^{n} \sigma_{n,k}^\kappa f_{n,k} \hat{\psi}_{n,k}^\kappa.$$
Proof (sketch)

• Take the functions in the range of $l_0$, namely,

$$\psi_{n,k}^\kappa, \quad n \geq 0, \quad 0 \leq k \leq n,$$

and prove that $Z_{n,k}^\kappa := l_0^* \psi_{n,k}^\kappa$ is orthogonal on $M$ for some weight [Maass, Louis].

• Also show that $l_0^* \psi_{n,k}^\kappa = 0$ for $k \notin 0 \ldots n$.

Note: $l_0^*$ depends on the weight in data space. Since $\psi_{n,k}^\kappa$ is orthogonal in $L^2(\partial^+ \Sigma M)$, it is natural to define $l_0^*$ w.r.t. this topology.
Proof (ugly)

\((\beta_-, \alpha_-)(\rho, \theta)\): coordinates of the unique curve through \((\rho e^{i0}, \theta)\).

\[
I^* \psi_{n,k}^\kappa (\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{S^1} e^{i(n-2k)(\beta_- + s(\alpha_-))} \frac{e^{i(n+1)s(\alpha_-)} + (-1)^n e^{-i(n+1)s(\alpha_-)}}{2 \cos(\alpha_-)} d\theta \\
\propto e^{i(n-2k)\omega} \int_{S^1} e^{i(n-2k)(\beta_- + s(\alpha_-))} U_n(\sin(s(\alpha_-))) s'(\alpha_-) d\theta \quad (U_n : \text{Cheb 2})
\]

Note the following relation:

\[
\beta_- (\rho, \theta) + s(\alpha_- (\rho, \theta)) + \pi = \theta - \tan^{-1} \left( \frac{\kappa \rho^2 \sin(2\theta)}{1 + \kappa \rho^2 \cos(2\theta)} \right) = \theta'(\rho, \theta)
\]

Now change variable \(\theta \to \theta'\) with \(\frac{\partial \theta'}{\partial \theta} \propto \frac{1 - \kappa \rho^2}{1 + \kappa \rho^2} s'(\alpha_- (\rho, \theta)):\n
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\]

Now use Euclidean knowledge.

At least 3 miracles along the way.
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The X-ray transform on constant curvature disks

**Const. Curv. Disks: Singular Value Decomposition**

**Visualization:** \( \{ Z_{n,k}^\kappa \}_{0 \leq n \leq 5, 0 \leq k \leq n}, \kappa = -0.8 \)
The X-ray transform on constant curvature disks

Const. Curv. Disks: Singular Value Decomposition

Visualization: \( \{ Z_{n,k}^{\kappa} \}_{0 \leq n \leq 5, 0 \leq k \leq n}, \kappa = -0.4 \)
The X-ray transform on constant curvature disks

Const. Curv. Disks: Singular Value Decomposition

Visualization: \( \{ Z_{n,k}^\kappa \} \), \( 0 \leq n \leq 5, 0 \leq k \leq n, \kappa = 0 \)
The X-ray transform on constant curvature disks

 Const. Curv. Disks: Singular Value Decomposition

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Conclusions

On the geodesic X-ray transform on constant curvature disks

- Range characterizations via either projection operators or moment conditions.
- SVD of $l_0$ for a special choice of weights on $M$ and $\partial_+ SM$.

Perspectives:
- tensor tomography, regularity of special invariant distributions,
- sharp Sobolev mapping properties for $l_0$.
- generalize to other (non-CC, non-symmetric) geometries.

Thank you

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