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Radon Transforms, Differential Equations, and Microlocal Analysis

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ABSTRACT. This article is an expanded version of my talk at the 2000 AMS/-IMS/SIAM conference on Radon transforms and Tomography. The goal is to show how microlocal analysis can be used in integral geometry and to show how uniqueness theorems for Radon transforms (proven using microlocal analysis) can be applied to PDE. Microlocal analysis is used to prove a support theorem of Boman and Quinto for the Radon transform on lines. A uniqueness theorem for the circle transform of Agranovsky and Quinto is described and applied to the wave equation.

1. Introduction

This is an expanded version of an introductory talk I gave at the AMS/IMS/-SIAM conference at Mt. Holyoke College on June 22, 2000. This reports joint work with Jan Boman [**9**] and with Mark Agranovsky [**2, 3**].

As one of the conference organizers, I want to thank all the participants for coming! I was happy to see several participants of this conference who also attended the 1993 conference on Tomography and Integral Geometry when they were students.

Radon transforms are integral transforms that integrate functions over sets. The most popular example is the line transform that integrates a plane function over lines. One can generalize this transform in many ways. For example, one can integrate with respect to different weights on the line as we discuss in §2 (see (2.1)). One can also integrate with respect to different curves, such as circles, which we consider in §3.

The basic questions are whether the transform is injective (and if so to find inversion formulas) range theorems, and support theorems. One uses a support theorem to infer support restrictions on a function f from support restrictions on its Radon transform, Rf. The classical support theorem [**22**] says that if all line

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integrals of a compactly supported continuous function $f \in C_c(\mathbb{R}^2)$ are zero for lines lying outside a convex set, the f is zero outside of the convex set.

Radon transforms can be applied to solve partial differential equations. For example, the classical Radon transforms reduces the wave equation in $\mathbb{R}^n$ to the one-dimensional wave equation (with a parameter) (e.g., [**24**]). One can use this reduction and properties of the Radon transform to understand solutions of the wave equation.

The goal of this paper is to show some of the ways microlocal analysis can be used in integral geometry and to demonstrate how integral geometry is applied to other areas in mathematics. We use microlocal analysis to prove a support theorem for the line transform [**9**]. We will apply a uniqueness theorem for the Radon transform on spheres [**2**] (that was proven using microlocal analysis) to derive properties of solutions of the wave equation.

Many researchers have used microlocal analysis to solve problems in integral geometry and tomography. Victor Guillemin and Shlomo Sternberg started it all when they showed that the Radon transform is a special type of Fourier integral operator (FIO), a so-called conormal distribution (since the Lagrangian manifold for this FIO is a conormal bundle [**18, 19**]). The geometric properties of this Lagrangian manifold determine the microlocal properties of the Radon transform. For example, the Lagrangian for the hyperplane transform we discuss in §2 has simpler geometric properties than the Lagrangian for the circle transform in §3. So, the proofs for this transform are easier.

Guillemin and Sternberg also used microlocal analysis to prove general range characterizations for Radon transforms [**20**]

D.H. Phong and Elias Stein [**31**], Allan Greenleaf and Gunther Uhlmann and others have used microlocal analysis to understand mapping properties of singular and nonsingular Radon transforms. Greenleaf and Uhlmann developed the microlocal theory for the geodesic line transform on admissible complexes [**15**]. They have recently proven a connection between uniqueness for the Dirichlet-to-Neumann problem and support theorems for the two-plane transform. They have used this to give conditions when a density is determined by limited Dirichlet-to-Neumann data [**16**].

Other researchers [**2, 9, 10, 14, 17, 35**] have used microlocal analysis to prove support theorems for Radon transforms with real-analytic weights. Microlocal analysis allows us to infer smoothness of a function from smoothness of its Radon transform. The proper concept of non-smoothness is the analytic wavefront set, Definition 2.3. These authors use the microlocal properties of the specific Radon transform to tell what smoothness of f is reflected in smoothness of its Radon transform, e.g., at points where when $Rf = 0$. Then a powerful theorem of Kawai, Kashiwara, and Hörmander, Theorem 2.6, is used to infer that f is zero if f is smooth in certain directions at points on the boundary of its support (i.e., those directions are not in the analytic wavefront set of f). We'll see how this procedure works for the Radon transform on lines in §2.

Microlocal analysis has been used in tomography to understand which singularities of a function are stably reconstructed from limited Radon data. A correspondence is worked out between singularities of f and singularities of Rf where R is the classical Radon transform on lines. Sobolev spaces provide an L^2 grading of smoothness, and Sobolev wavefront sets tell the smoothness of singularities in this grading. In [**34**] one sees how this smoothness is reflected in the smoothness of singularities in the data. This was generalized to lambda CT and the attenuated Radon transform in [**29**].

David Finch [**13**] and Alexander Katsevich [**27**] have taken the results in [**15, 34**] farther by using microlocal analysis to understand artifacts coming from algorithms in three-dimensional X-ray CT. Microlocal ideas have been applied to singularity detection in sonar by Louis and Quinto [**30**].

In Section 2, we will use microlocal ideas to prove a support theorem for the Radon transform on lines, and in Section 3 we will see a more complicated uniqueness theorem for a Radon transform on circles. This proof requires not only microlocal analysis but also a relationship between the Radon transform and harmonic polynomials. This result will be applied to learn qualitative properties of solutions to the wave equation in section 3.3.

2. A Support Theorem for the Radon transforms on lines in the plane

The Radon transform on lines models X-ray computed tomography, and it is one of the most fundamental transforms in integral geometry. Let Ξ be the set of lines in the plane and let μ be a nowhere zero real-analytic function on $\mathbb{R}^2 \times \Xi$. We define the Radon transform of $f \in C_c(\mathbb{R}^2)$

$$R_\mu f(\ell) = \int_{x\in\ell} f(x)\mu(x,\ell)ds \tag{2.1}$$

The classical Radon transform on lines is (2.1) with $\mu = 1$. Inversion formulas, range theorems, and support theorems are known for this classical transform (e.g., [**22, 23**]).

Transforms with general weights, $\mu \neq 1$, come up in emission tomography (the attenuation of the body adds an exponential weight in (2.1)) and in other applications.

One theme of my research has been how properties of a Radon transform depend on the weight. In 1980, using microlocal analysis, I showed the relationship between the weight for a Radon transform (such as the line transform) and whether the transform was invertible by a differential operator. Then, I proved that nonzero rotation invariant transforms on hyperplanes were invertible and satisfied a support theorem [**32**]. This early result made me believe that Radon transforms on lines would to be invertible as long as the weight μ is smooth and positive.

However, in 1984 Jan Boman constructed a very important example of a positive C^∞ weight μ and a function $f \in C_c^\infty(\mathbb{R}^2)$, $f \neq 0$, such that $R_\mu f = 0$ [**8**]. This demonstrated that the relationship between invertibility and the weight is extremely subtle. He proved invertibility if the weight is real-analytic and nowhere zero [**7**]. Together, we proved Theorem 2.1 and then support theorems for the line transform integrating over lines through a real-analytic curve (or lines tangent to a surface) in $\mathbb{R}^3$ [**10**].

THEOREM 2.1 ([**9**]). *Let $\mu(x,\ell)$ be a nowhere zero real-analytic function on $\mathbb{R}^2 \times \Xi$. Let $f \in \mathcal{E}'(\mathbb{R}^2)$. Let $\mathcal{A} \subset \Xi$ be an open connected set of lines. Assume some $\ell_0 \in \mathcal{A}$ is disjoint from* $\operatorname{supp} f$ *and assume $R_\mu f(\ell) = 0$ $\forall \ell \in \mathcal{A}$. Then, all lines in $\mathcal{A}$ are disjoint from* $\operatorname{supp} f$.

The point is that, if all line integrals of f are zero and if f is zero near some line in $\mathcal{A}$, then one can "analytically continue" f to be zero near all lines in $\mathcal{A}$.

The theorem is true for a large class of Radon transforms including the hyperplane transform [**33**]. These transforms all satisfy the Bolker Assumption, a geometric assumption on the Lagrangian manifold associated to the Radon transform [**18, 19**]. For the classical transform Theorem 2.1 follows from Helgason's support theorem ([**24**], Lemma 2.11).

EXAMPLE 2.2. *What support restrictions do we get on* $f \in C_c(\mathbb{R}^2)$ *if* $R_\mu f(\ell) = 0$ *for all lines in the sets below*

(1) $\mathcal{A}$ *is the set of all lines* ℓ *of angle within one degree of the* $x-$*axis?*
(2) $\mathcal{B}$ *is the set of lines not meeting* $W = [0,1]^2 \cup ([3,4] \times [0,1])$*?*

The set of lines in (1), $\mathcal{A}$, is open and connected, and it fills $\mathbb{R}^2$. Since $f \in C_c(\mathbb{R}^2)$, some line in this set is disjoint from $\operatorname{supp} f$. Therefore Theorem 2.1 can be used to show $f = 0$ on all lines in $\mathcal{A}$, that is $f \equiv 0$ on $\mathbb{R}^2$. The set of lines, $\mathcal{B}$, in (2) has two connected components, one consisting of the lines outside the convex hull of W and the other consisting of all lines between the two squares in W. Since f has compact support, Theorem 2.1 implies that f is zero outside of the convex hull of W. However, the theorem says nothing about the part of $\operatorname{supp} f$ between the squares. If we know that f is zero near one line between the squares, then $f \equiv 0$ outside of W by Theorem 2.1. Can you construct a counterexample of a continuous function f that is supported in the convex hull of W and for which $R_1 f(\ell) = 0$ for all lines in $\mathcal{B}$? (HINT: Think about thin strips with value $+1$ and -1.)

PROOF OF THEOREM 2.1. There are three key ideas. First, we define microlocal analytic singularity: the analytic wavefront set. Second, we give a theorem that determines what microlocal singularities R_μ detects, and finally, we quote a strong theorem of Kawai, Kashiwara, and Hörmander [**25**] on analytic wavefront at the boundary of the support of a function.

Real-analytic wavefront sets allow us to characterize singularities in the real-analytic category. Let $\mathcal{F}f(\xi) = \int_{x\in\mathbb{R}^2} f(x)e^{-ix\cdot\xi}dx$ be the Fourier transform of $f \in L^1(\mathbb{R}^2)$. If

$$\forall \xi \in \mathbb{R}^2 \quad |\mathcal{F}f(\xi)| \le Ce^{-c|\xi|} \quad \text{for some } C > 0 \text{ and } c > 0, \tag{2.2}$$

then f is real-analytic since f can be extended to be holomorphic in a strip in $\mathbb{C}^2$, $\{x + iw \in \mathbb{C}^2 \mid |w| < c\}$:

$$f(x + iw) = \frac{1}{(2\pi)^2} \int_{\xi\in\mathbb{R}^2} \mathcal{F}f(\xi)e^{i(x+iw)\cdot\xi}d\xi\,. \tag{2.3}$$

If $|w| < c$, the integral (2.3) converges because $\mathcal{F}f$ decreases more quickly by (2.2) than $e^{i(x+iw)\cdot\xi}$ increases. Since $e^{i(x+iw)\cdot\xi}$ is holomorphic in $z = x + iw$, (2.3) gives f as the restriction of a function that is holomorphic on this strip.

Formula (2.2) relates global exponential decrease of $\mathcal{F}f$ with global real-analyticity of f.

To localize this idea, we need a real-analytic cut off function. Of course, there is a problem because any real-analytic function that is zero on an open set is zero everywhere. So, we use a Gaussian as an almost-cutoff function–we ask whether $\mathcal{F}f$ times a Gaussian (that becomes more localized as a limit is taken) is exponentially decreasing.

However, we don't just want to know where f is real-analytic, but we want to try to find directions in which f is real-analytic, locally. We use the Fourier-Bros-Iagolnitzer transform (2.4) ([**25**] Ch. 8).

DEFINITION 2.3. Let $f \in \mathcal{E}'(\mathbb{R}^2)$ and let $x_1 \in \mathbb{R}^2$ and $\xi_1 \in \mathbb{R}^2 \setminus 0$. Then, $(x_1, \xi_1) \notin \mathrm{WF}_A(f)$ **if and only if** there are neighborhoods U of x_1 and V of ξ_1 and $\exists C > 0,\ \exists c > 0$ such that for all $x \in U,\ \xi \in V$:

$$\left| \int_{y\in\mathbb{R}^2} \left(f(y)e^{-\lambda|x-y|^2}\right) e^{-iy\cdot(\lambda\xi)} dy \right| \le Ce^{-c\lambda}. \tag{2.4}$$

The function f is localized as Gaussian becomes more focused at y as $\lambda \to \infty$. The exponential decrease of the localized Fourier transform is given by the inequality in (2.4). And the direction (microlocalization) is represented by the requirement that the Fourier transform exponentially decreases in directions near the vector ξ_1: $\lambda\xi$ with ξ near ξ_1 as $\lambda \to \infty$.

EXAMPLE 2.4. *Calculate* $\mathrm{WF}_A(f)$*, if f is the characteristic function of the lower half plane in* $\mathbb{R}^2$. If you calculate the FBI transform of f, (2.4) you will see it is exponentially decreasing for all points not on the x-axis. For points on the $x-$axis, the only directions in which it is not exponentially decreasing are those in vertical directions. That is, $\mathrm{WF}_A(f)$ is the conormal bundle of the x-axis, the set of all covectors perpendicular to the x-axis. In a similar way, one can show that if f is the characteristic function of a disk in the plane, then the analytic wavefront set of f is the conormal bundle of the boundary circle, the set of nonzero vectors perpendicular to the boundary.

Note that we will identify covectors (in $T^*\mathbb{R}^n$) with vectors (in $T\mathbb{R}^n$) and conormal covectors to a set with normal vectors to that set.

Now, we look at how the Radon transform (2.1) detects analytic wavefront set. In general, Radon transforms detect singularities perpendicular to the sets being integrated over but not in other directions. This makes sense from elementary considerations, even though the proof requires microlocal analysis, in general. Let's look at a simple example. Let f be the characteristic function of the unit disk, then we know $\mathrm{WF}_A(f)$ consists of $\{(x, cx) \mid |x| = 1, c \in \mathbb{R}, c \neq 0\}$, the normals at the boundary by Example 2.4. Furthermore, $R_1 f(p) = 2\sqrt{1-p^2}$ where p is the distance from the line to the origin. This function is real-analytic except when $p = \pm 1$, that is except for all lines tangent to the boundary of the unit disk. These lines have normals that are normal to the boundary, i.e., the normals to the lines at points of intersection with $|x| = 1$ correspond to the singularities of f that are detected by the Radon data. In fact, a more subtle version of this observation and its converse are true in general.

THEOREM 2.5 (Microlocal Regularity Theorem, [**9**]). *Let $f \in \mathcal{E}'(\mathbb{R}^2)$. Let μ be a nowhere zero real-analytic function on $\mathbb{R}^2 \times \Xi$ and let R_μ be the associated Radon transform on lines. Let $\ell_1 \in \Xi$, and let ξ_1 be conormal to ℓ_1. If $R_\mu f$ is zero near ℓ_1, then $(x_1, \xi_1) \notin \mathrm{WF}_A(f)$ $\forall x_1 \in \ell_1$.*

Morally, this is related to the projection slice theorem (the relation between R_1 and $\mathcal{F}$). The reason is that R_μ is an elliptic Fourier integral operator associated to a specific conormal bundle, and R_μ detects singularities conormal to the line being integrated over but not other singularities. So, if $R_\mu f$ is smooth near a line ℓ_1, then f is smooth at all points on ℓ_1, at least in the direction conormal to ℓ_1.

An important theorem of Kawai, Kashiwara and Hörmander gives precise information about analytic wavefront set at $\operatorname{bd}\operatorname{supp} f$.

THEOREM 2.6 (Microlocal Holmgren Theorem, [**25**] Theorem 8.5.6). *Let $f \in \mathcal{D}'(\mathbb{R}^2)$. Let ℓ_1 be a line that meets* $\operatorname{supp} f$ *and such that* $\operatorname{supp} f$ *is on one side of* ℓ_1. *Let $x_1 \in \ell_1 \cap (\operatorname{bd}\operatorname{supp} f)$ and let ξ_1 be conormal to ℓ_1. Then, $(x_1, \xi_1) \in \mathrm{WF}_A(f)$.*

We know f cannot be real-analytic at x_1 since x_1 is a boundary point of $\operatorname{supp} f$. This theorem gives the stronger result that the conormal wavefront direction must be in $\mathrm{WF}_A(f)$.

Now, we prove the theorem by contradiction. Assume there is a line $\ell_2 \in \mathcal{A}$ that meets $\operatorname{supp} f$. Then, by connectedness, there is a line $\ell_1 \in \mathcal{A}$ such that ℓ_1 touches $\operatorname{supp} f$ and $\operatorname{supp} f$ is on one side of ℓ_1. To see this, imagine moving the line ℓ_0 on a path through $\mathcal{A}$ to the line ℓ_2; ℓ_0 does not meet $\operatorname{supp} f$, but ℓ_2 does. So, there must be a first line in the path, ℓ_1, that meets $\operatorname{supp} f$.

Let $x_1 \in (\operatorname{supp} f \cap \ell_1)$ and let ξ_1 be conormal to ℓ_1.

By Theorem 2.6 and because $\operatorname{supp} f$ is on one side of ℓ_1, $(x_1, \xi_1) \in \mathrm{WF}_A(f)$.

However, by Theorem 2.5 and because $R_\mu f = 0$ near ℓ_1, $(x_1, \xi_1) \notin \mathrm{WF}_A(f)$.

This contradiction proves the theorem. A lose summary of the proof is as follows. Theorem 2.6 shows that if a line, ℓ_1, in $\mathcal{A}$ is tangent to the boundary of $\operatorname{supp} f$, then f has wavefront set normal to ℓ_1. But Theorem 2.5 shows that f does not have any wavefront set in directions normal to lines in $\mathcal{A}$. Therefore, no line in $\mathcal{A}$ can be tangent to the boundary of $\operatorname{supp} f$. Since f is zero near ℓ_0 and $\mathcal{A}$ is connected, no line in $\mathcal{A}$ can meet $\operatorname{supp} f$. □

3. The Radon transform on circles in the plane

Now, we investigate a circular transform and use it to prove properties of zero sets of solutions to the wave equation.

Let $a \in \mathbb{R}^2$ and let $r > 0$. Let $C(a, r)$ be the circle centered at a and with radius r. We define the circular Radon transform of $f \in C_c(\mathbb{R}^2)$

$$Rf(a,r) = \int_{x \in C(a,r)} f(x)\,ds \tag{3.1}$$

This transform has a rich theory. If integrals of $f \in C(\mathbb{R}^2)$ are given over all circles, then an elementary inversion formula is given by $f(a) = \lim_{r \to 0^+} \frac{1}{2\pi r} Rf(a, r)$. Inversion is easy because this transform is geometrically overdetermined. The set of all circles in the plane has dimension 3 (2 for $a \in \mathbb{R}^2$ plus 1 for $r \in (0, \infty)$), but the plane has dimension two. Injectivity holds even when the radii are bounded away from zero [**38**].

Several authors have used microlocal techniques in this area. In [**17**], the authors prove a fairly general support theorem on a real-analytic manifold, M.[1]

So, it is natural to ask when will injectivity hold with restricted sets of centers and radii?

First, let's explore what happens when we fix the radius r_0 but let the centers vary in an open set in $\mathbb{R}^2$. If $f \in C_c(\mathbb{R}^2)$, then $Rf(a, r_0) = f * \mathcal{M}$ where $\mathcal{M}$ is the measure on the sphere $S(0, r_0)$. If we take Fourier transforms, we get that

[1]Let $S(a, r)$ be the geodesic sphere of radius r centered at $a \in M$. Assume $\mathcal{A} \subset M \times (0, I_M)$ is an open connected set. Here I_M is the injectivity radius of the manifold M. This says that any sphere on M of radius less than I_M is diffeomorphic to a Euclidean sphere, and its interior is diffeomorphic to an open disk in Euclidean space. See [**28**] for more information. If $Rf(a, r) = 0$ for all $(a, r) \in \mathcal{A}$ and if for some $(a_0, r_0) \in \mathcal{A}$, the sphere $S(a, r)$ is disjoint from $\operatorname{supp} f$, then $f = 0$ on the union of spheres in $\mathcal{A}$.

$\mathcal{F}(f * \mathcal{M})$ is the product of $\mathcal{F}f$ and $\mathcal{F}\mathcal{M}$. The function $\mathcal{F}\mathcal{M}$ is a Bessel function [**39**]. If $Rf(a, r_0) = 0,\ \forall a \in \mathbb{R}^2$, then this argument shows that $\mathcal{F}f$ must be zero almost everywhere because the Bessel function $\mathcal{F}\mathcal{M}$ is nonzero almost everywhere. Therefore, f is zero.

However, there are functions, f, not of compact support such that $Rf(a, r_0) \equiv 0$ [**39**]. On the other hand, one can show that data $Rf(a, r_0)$ for $a \in \mathbb{R}^2$ determines f, if f is assumed to be zero on a neighborhood of one disk of radius r_0 [**26**], (see [**35**] for real-analytic manifolds).

In general, integrals over spheres of one radius do not determine f, but integrals over spheres of two "well chosen" radii determine $f \in C(\mathbb{R}^n)$ [**12, 38, 39**]. Berenstein and Zalcman generalized this to symmetric spaces of real rank one [**6**]. Properties including mean value theorems, injectivity, and inversion formulas were proven by Asgeirsson and John [**26**] for $\mathbb{R}^n$, and by Helgason [**21**] for homogeneous spaces. Local inversion formulas and support theorems are known for transforms with standard weights on disks in symmetric spaces [**37**], on disks of two radii in $\mathbb{R}^n$ [**5**]. Schneider uses spherical harmonics to prove injectivity of a large class of spherical transforms on the sphere, [**36**]. These results are, in general, proven using the harmonic analysis of the specific ambient spaces. Microlocal techniques have been used to prove specialized two-radius theorems for spherical transforms on $\mathbb{R}^n$ and real-analytic manifolds for transforms with arbitrary nowhere zero real-analytic weights [**41, 42**]. The article [**39**] is a wonderful introduction to the spherical and related transforms, and [**40**] and [**4**] provide comprehensive bibliographies and summaries of results. Larry Zalcman's article in these proceedings updates these bibliographies and provides valuable perspective.

Now, let's examine what happens in the plane when we restrict the set of centers but allow the radii to be arbitrary for the classical circular transform, (3.1).

DEFINITION 3.1. Let $S \subset \mathbb{R}^2$. S is called a *set of injectivity* for R **if and only if** $Rf(a, r) = 0,\ \forall a \in S,\ \forall r > 0$ implies $f = 0$ for all $f \in C_c(\mathbb{R}^2)$. If S is not a set of injectivity, S is called a set of noninjectivity.

EXAMPLE 3.2. *Are the following sets of injectivity?*

(1) $S = \mathbb{R}^2$
(2) $S =$ *one point?*
(3) $S =$ *a line?*
(4) $S =$ *two parallel lines?*

We have already seen that $S = \mathbb{R}^2$ is a set of injectivity (any continuous function f is determined by integrals over all circles). If S is one point, we would guess S is not a set of injectivity. In fact, radial functions about that point map surjectively onto the range, so one point is not a set of injectivity. $S =$ a finite set is also not a set of injectivity. The case of a line is more interesting. The set S is one-dimensional, so we have data Rf over a two dimensional set of circles. Courant and Hilbert proved the following theorem.

THEOREM 3.3 ([**11**]). *If ℓ is a line, then ℓ is not a set of injectivity even for arbitrary continuous functions. The null space for R is the set of continuous functions that are odd about this line.*

We can use Theorem 3.3 to answer item 4. Let $S = \ell_1 \cup \ell_2$ be two parallel lines. Let f have compact support and assume Rf is zero for all centers on S. Then, f is odd about both parallel lines, and f has compact support. By successively reflecting f in both lines, we see f must be zero because f has compact support.

3.1. Relation of R to Harmonic Polynomials. An important way to learn more about sets of injectivity for R is to develop their relationship to harmonic polynomials.

DEFINITION 3.4. For $f \in C_c(\mathbb{R}^2)$ and $k \in \{0, 1, 2, ...\}$ let

$$Q_k[f] = f * |x|^{2k}, \qquad S[f] = \{a \in \mathbb{R}^2 \mid Rf(a,r) = 0, \ \forall r > 0\}. \tag{3.2}$$

Notice that $Q_k[f]$ is a polynomial of degree at most $2k$ since $|x|^{2k}$ is such a polynomial. The set $S[f]$ is the set of centers at which f has zero circular integrals: the set of non-injectivity for f.

THEOREM 3.5. *For $f \in C_c(\mathbb{R}^2)$,*

(1) $S[f] = \bigcap_{k=0}^{\infty} V(Q_k[f])$ *where $V(Q)$ is the zero set of the polynomial Q.*
(2) *Assume $f \not\equiv 0$. Then, $Q_k[f] \neq 0$ for some k. If k_0 is the smallest integer such that $Q_k[f] \neq 0$, then $Q_{k_0}[f] = f * |x|^{2k_0}$ is a harmonic polynomial.*

If $f \not\equiv 0$, then Theorem 3.5 demonstrates that $S[f]$ is a subset of the zero set of the nontrivial harmonic polynomial, $Q_{k_0}[f]$. This was first proved by Lin and Zobin.

PROOF OF THEOREM 3.5. By definition

$$a \in S[f] \quad \textbf{if and only if} \quad \int_{|x-a|=r} f(x)ds = 0 \ \text{ for all } \ r > 0.$$

If we integrate with respect to r^{2k} from 0 to ∞, using polar coordinates, then we get

$$\begin{aligned} \int_0^\infty Rf(a,r) r^{2k} dr &= \int_0^\infty \left(\int_{|x-a|=r} f(x)ds \right) |x-a|^{2k}\, dr \\ &= f * |x|^{2k}(a) = Q_k[f](a) \end{aligned} \tag{3.3}$$

Since f has compact support, $Rf(a,r) = 0$ for all r **if and only if** all these moments $Q_k[f](a) \equiv 0$. Thus, (1) holds.

If $Q_k[f] = f * |x|^{2k} \equiv 0$ for all k, then f is zero because Theorem 3.5, (1) shows $Rf(a,r) = 0$ for all (a,r). So, if $f \not\equiv 0$, then some $Q_k[f]$ is nonzero. Let k_0 be the smallest index such that $Q_k[f] \not\equiv 0$. To prove (2), we just note that

$$\Delta Q_{k_0}[f] = f * (\Delta |x|^{2k_0}) = \text{const} f * |x|^{2k_0 - 2} = 0$$

as k_0 is the smallest integer, k such that $Q_k[f] \neq 0$. Therefore $Q_{k_0 - 1}[f] = 0$ and $Q_{k_0}[f]$ is harmonic. □

Theorem 3.5 allows to get better information about sets of injectivity.

EXAMPLE 3.6. *Is an equilateral triangle, T, a set of injectivity?* We prove this by contradiction. Assume f is a nonzero function such that $T \subset S[f]$. Let $Q_{k_0}[f]$ be the nontrivial harmonic polynomial in Theorem 3.5 (2). By this theorem, T is a subset of the zero set of $Q_{k_0}[f]$. By the maximum principle, the zero set of a nontrivial harmonic polynomial contains no closed curve. This contradiction shows T is a set of injectivity. In fact, this proof shows that every closed curve a set of injectivity.

3.2. Characterization of sets of injectivity. The following sets will be the basis of our sets of noninjectivity.

DEFINITION 3.7. The *Coxeter Set* Σ_N $(N \in \mathbb{N})$ is the set of lines through the origin and N^{th} roots of unity. We define $\Sigma_0 = \emptyset$.

One can show any Coxeter system is not a set of injectivity. One just chooses a function compactly supported in one of the "Vs" of the system and reflects it oddly about each of the lines in the system. The resulting function is odd about each of the lines. Our next theorem shows that Coxeter systems are essentially the only sets of noninjectivity.

THEOREM 3.8 ([**2**]). *The condition*

$$\begin{aligned}&\text{the set } S \text{ is not contained in any set of the form}\\ &k(\Sigma_N) \cup F, \text{ where } k \text{ is a rigid motion and } F \text{ is a finite set}\end{aligned} \tag{3.4}$$

is necessary and sufficient for S to be a set of injectivity for the Radon transform over circles.

EXAMPLE 3.9. *Are the following sets of injectivity?*

(1) *A small "curved" curve?*
(2) *An* L *(right angle)?*
(3) *A* W*?*

According to Theorem 3.8, any set of non-injectivity must contain lines. So, a curved curve is a set of injectivity. An L is not a set of injectivity since it is a subset of some rigid motion of Σ_2. The segments in a W do not meet at a single point, so they do form a set of injectivity.

Theorem 3.8 demonstrates that lines (and finite points) are the basic building blocks of all sets of non-injectivity. The only sets of non-injectivity are the examples we saw, very symmetric unions of lines and finite sets. In fact, we show [**2**] that for each $f \in C_c(\mathbb{R}^2)$,

$$\text{if } S[f] \neq \emptyset, \text{ then } S[f] = k(\Sigma_N) \cup F \text{ for some } N = 0, 1, 2, \dots \tag{3.5}$$

where k is a rigid motion of $\mathbb{R}^2$ and F is a finite set. This implies that if $S[f]$ is an infinite set, $S[f]$ is the zero set of a homogeneous harmonic polynomial (after translation) union a finite set (because these are the sets in (3.5)).

The proof of Theorem 3.8 is done in steps. We let $f \neq 0$ and assume $S[f]$ is an infinite set. By Theorem 3.5, we know that $S[f]$ is a subset of the zero set of a harmonic polynomial. If this polynomial is not homogeneous, we prove that this subset, $S[f]$, must satisfy a specific geometric condition. Then, we prove using microlocal analysis that any set satisfying this condition is a set of injectivity. This shows $f = 0$ or (3.4) holds.

The microlocal analysis of the circular transform is more difficult than for the line transform. For example, the microlocal regularity theorem for this transform that corresponds to Theorem 2.5 is more complicated geometrically ([**2**], Lemma 4.3). So, we must use not only microlocal analysis but also the relation of R to harmonic polynomials to prove the uniqueness theorem.

For functions not of compact support, injectivity sets are harder to characterize. For example, concentric circles can be sets of non-injectivity.

Sets of non-injectivity have been characterized in $\mathbb{R}^n$ in special cases [**1, 3**], but the analogue of Theorem 3.8 has only been conjectured [**2**]. The problem is that the geometry and harmonic analysis are much more complicated in $\mathbb{R}^n$ than in the plane.

3.3. Applications to the wave equation. Consider the wave equation IVP:

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial t^2} \\ u(x,0) = 0, \qquad & \frac{\partial}{\partial t} u(x,0) = f(x), \ f \in C_c(\mathbb{R}^2) \end{aligned} \tag{3.6}$$

DEFINITION 3.10. The *stationary set* for the IVP for f, (3.6), is the set

$$N[f] = \{a \in \mathbb{R}^2 \mid u(a,t) = 0 \ \forall t > 0\}$$

Our goal is to characterize stationary sets when $f \in C_c(\mathbb{R}^2)$. By the Poisson-Kirchoff formula, the solution to (3.6) satisfies:

$$\begin{aligned} u(a,t) &= \frac{1}{t} \int_{|a-\xi| \le t} \frac{f(\xi) d\xi}{\sqrt{t^2 - |a-\xi|^2}} \\ &= \frac{1}{t} \int_0^t \frac{Rf(a,r) dr}{\sqrt{t^2 - r^2}} \end{aligned} \tag{3.7}$$

But, (3.7) is an invertible Abel equation of the first kind. Therefore,

$$Rf(a,r) = 0 \ \forall r > 0 \ \textbf{if and only if} \ u(a,t) = 0 \ \forall t > 0\,.$$

This shows that

$$S[f] = N[f] \tag{3.8}$$

and (3.5) and (3.8) gives us the theorem:

THEOREM 3.11 ([**2**]). *Let $f \in C_c(\mathbb{R}^2)$ and let u be the solution to the IVP (3.6). If $N[f] \neq \emptyset$, then for some $M = 0, 1, 2, \ldots$,*

$$N[f] = k(\Sigma_M) \cup F$$

where k is a rigid motion and F is a finite set.

The correspondence (3.8) is valid in $\mathbb{R}^n$ and for arbitrary continuous functions. Mark Agranovsky and I have used this to generalize Theorem 3.11 to the IVP (3.6) in $\mathbb{R}^n$ when the initial data, f, is a sum of point distributions [**3**] and we are working on the Dirichlet problem on nice domains in $\mathbb{R}^2$.

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