SUPPORT THEOREMS FOR THE SPHERICAL RADON TRANSFORM ON MANIFOLDS

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Abstract. Let M be a real-analytic manifold and let $S \subset M$ be a real-analytic hypersurface. We prove local support theorems for the spherical Radon transform that integrates over spheres centered at points on S. Our theorems are valid for a broad class of distributions with initial support restrictions depending on S and the injectivity radius of M. The proofs involve the microlocal analysis of the sphere transform and a microlocal Holmgren theorem of Kawai, Kashiwara, and Hörmander.

1. Introduction

In this article, we will prove local support theorems for the Radon transform that integrates over spheres with center on a real-analytic hypersurface S in a connected real-analytic Riemannian manifold M. Our theorems are valid for a broad class of distributions with initial support restrictions depending on S and the injectivity radius of M. The proofs involve the microlocal analysis of the sphere transform and a microlocal Holmgren theorem of Kawai, Kashiwara, and Hörmander. This transform has been well studied in $\mathbb{R}^n$, but less is known for spherical transforms on manifolds.

For $y \in M$ we define the sphere and disks centered at $y \in M$ and of radius $r > 0$ as follows

$$\begin{gathered} S(y,r) = \{x \in M \mid d(x,y) = r\} \qquad D(y,r) = \{x \in M \mid d(x,y) < r\}, \\ \overline{D}(y,r) = \operatorname{cl} D(y,r) \end{gathered} \tag{1.1}$$

where $d(\cdot,\cdot)$ is the geodesic distance. If $y \in M$, then we let I_y be the *injectivity radius of M at y*.[1]

Date: February, 2006.

2000 *Mathematics Subject Classification.* Primary: 44A12, Secondary: 58C35.

Key words and phrases. Spherical Radon Transform, Microlocal Analysis, Support Theorem.

The author thanks Mark Agranovsky, Carlos Berenstein, Jan Boman, Sigurdur Helgason, and Larry Zalcman for stimulating discussions over the years. He appreciates enlightening conversations with Fulton Gonzalez about differential geometry. The comments of the referee and editor were very helpful, and they improved the article. This research is based upon work supported by the National Science Foundation under grants DMS-0200788 and DMS-0456858 and support from Tufts University FRAC and the Gelbart Institute of Bar Ilan University.

[1]Let $\mathcal{V}$ be the largest connected open neighborhood of the zero section in the tangent space TM such that the map $\phi : \mathcal{V} \to M \times M$, $\phi(y, X) = (y, \exp_y X)$, is real-analytic (see [26, III 8.1]). For $y_0 \in M$ and $r > 0$, let $\mathcal{D}(0, r)$ be the open ball about $0 \in T_{y_0}M$ of radius r. Then, the *injectivity radius* I_{y_0} is the largest number s such that $\{y_0\} \times \mathcal{D}(0, s) \subset \mathcal{V}$ and $\exp_{y_0}$ is a diffeomorphism on $\mathcal{D}(0, s)$. From this definition, it is easy to see that ϕ is a diffeomorphism (in the second coordinate) on a neighborhood in $\mathcal{V}$ of $\{y_0\} \times \mathcal{D}(0, I_{y_0})$ and so I_y is a lower semicontinuous function for $y \in M$.

For $r < I_y$, the sphere $S(y,r)$ is diffeomorphic to a Euclidean sphere, and it is the boundary of $D(y,r)$ [26]. We let

$$N = \{(y,r) \in S \times (0,\infty) \,\big|\, \forall z \in \overline{D}(y,r), r < I_z)\} \tag{1.2}$$

represent this set of spheres we consider. We include the injectivity radius assumption so that the Radon transform and its dual are well defined. The *incidence relation* of points and spheres is

$$Z = \{(x,(y,r)) \,\big|\, (y,r) \in N,\ x \in S(y,r)\}\,. \tag{1.3}$$

Since $z \mapsto I_z$ is lower semicontinuous (see footnote 1) and $\overline{D}(y,r)$ is compact for each $y \in M$ and $r < I_y$, N is a manifold that is open in $S \times (0,\infty)$. Therefore, Z is a submanifold of $M \times N$. Now, we can define the *spherical Radon transform* for a continuous function, $f \in C(M)$

$$R_\mu f(y,r) = \int_{x \in S(y,r)} f(x)\mu(x,y,r)\, dM_r(x)\,, \tag{1.4}$$

where $\mu(x,y,r)$ is a nowhere zero real-analytic weight on Z and $dM_r(x)$ is the Riemannian measure on $S(y,r)$ inherited from the Riemannian measure, dM on M (for $r_0 < I_y$, $\int_{\overline{D}(y,r_0)} f(x)dM = \int_{r=0}^{r_0}\int_{x\in S(y,r)} f(x)dM_r(x)dr$). Now, let $g \in \mathcal{D}(N)$, then the dual operator is

$$R_\mu^* g(x) = \int_{y \in S} g(y,d(x,y))\mu(x,y,d(x,y))dM_r(y).$$

Note that $R_\mu^* g$ can be defined for all $x \in M$ because g has compact support in N and can be smoothly extended to all $S \times (0,\infty)$ by setting g to zero off of N. Furthermore, it is clear that $R_\mu^* : \mathcal{D}(N) \to \mathcal{D}(M)$ is continuous. Therefore $R_\mu : \mathcal{D}'(M) \to \mathcal{D}'(N)$ is defined and continuous in the weak topology.

The spherical transform in $\mathbb{R}^3$ is a model for sonar and thermoacoustic tomography (TCT) and a related transform is a model of radar [11]. In Section 3, we will prove local uniqueness theorems motivated by these applications.

Many uniqueness results are known for the spherical transform for $M = \mathbb{R}^n$. If $S = H$ is a hyperplane and the weight is the classical weight, $\mu = 1$, Courant and Hilbert [12, p. 699] showed that the null space of this transform is the set of all functions odd about H.

Inversion methods exist for this transform, including backprojection and Fourier methods [8, 15, 25, 29], integral equations methods [30], and a reduction to the classical transform [13, 33]. Range theorems are in [7, 29, 33]. Numerical algorithms have been developed and tested [14, 25, 38, 41]. Recently Aleksei Beltukov proposed a numerical inversion method using a discrete SVD for the sonar transform. He showed that the singular values are fairly flat and then drop off precipitously which reflects the ill-posedness of the problem.

Less is known if the hypersurface $S \subset \mathbb{R}^n$ is not a hyperplane. Integral equations type inversion formulas exist if S is a circle in the plane [31] and if S is a cylinder in $\mathbb{R}^3$ [32]. Agranovsky and the author [2, 3] completely characterized sets $S \subset \mathbb{R}^2$ on which the circle transform is injective for compactly supported functions (and $\mu = 1$). Injectivity results are known for functions not of compact support in the plane and $\mathbb{R}^n$ (e.g., [1, 4, 5, 6]) and range results are in [7]. Finch, Patch, and Rakesh proved an elegant, explicit inversion method [16] for recovering a function from spherical integrals when the center set, S, is the boundary of a bounded, smooth, connected, open set in $\mathbb{R}^n$. A general inversion method is given for this problem and functions not of compact support [36], and [34] provides an overview of the area.

The microlocal analysis of the spherical transform will be a key to our proofs. Louis and Quinto use the microlocal analysis of the transform in $\mathbb{R}^n$ to characterize singularities (jumps, etc.) of the object that are stably visible from sonar data. Palamodov [33] provides inversion methods and strong instability results using microlocal analysis when S is a hyperplane.

Stationary sets for the wave equation are points in $\mathbb{R}^n$ at which the solution to the IVP (with zero initial position and compactly supported initial velocity) is zero for all time. Agranovsky and the author use this microlocal analysis and properties of zero sets of harmonic polynomials to understand stationary sets for the wave equation [4]. They gave a complete characterization in $\mathbb{R}^2$ [3] and for the Dirichlet boundary value problem on crystallographic domains in $\mathbb{R}^n$ [5].

In the general case, when M is a manifold, less is known. A result of Zalcman and the author [3, Theorem 7.1] characterizes sets of injectivity for the case of geodesic spheres centered on a hypersurface on a sphere. Aleksei Beltukov has proved an inversion method for the spherical transform with standard measure on spheres centered on a geodesic hyperplane on hyperbolic space [9]. For spheres of a finite number of radii on symmetric spaces, Berenstein and Zalcman (e.g., [10]) have proved lovely local support theorems. Helgason (e.g., [19, 20, 21]) has done much important work on this transform. Volchkov (e.g., [40]) has proved Pompeiu theorems for disks on manifolds. Microlocal techniques have been used to prove theorems for spheres of one radius and arbitrary nowhere zero real-analytic μ [35] and for spheres of arbitrary radius and center [17].

2. The Main Result

To state our most general theorem we need some background material. Note that, in general, sets written in script, such as $\mathcal{B}, \mathcal{C}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{V}$, will be subsets of a tangent space.

Definition 2.1. Let M be a Riemannian manifold, S a hypersurface and $s_0 \in S$ with injectivity radius I_{s_0}. Let $r \in (0, I_{s_0})$ and let $\mathcal{D}(0,r)$ be the open disk in $T_{s_0}(M)$ centered at the origin and of radius r and let $\mathcal{S}(0,r)$ be the origin-centered sphere of radius r. Let $\exp_{s_0}$ be the exponential map from $T_{s_0}M$ to M. Let $\mathcal{T}$ be the Euclidean tangent plane to $\exp_{s_0}^{-1}(S)$ at $0 \in T_{s_0}M$. The *tangent manifold to S at s_0* is the hypersurface

$$T(s_0) = \exp_{s_0}\left(\mathcal{T} \cap \mathcal{D}(0, I_{s_0})\right).$$

Our next proposition gives a more intrinsic description of $T(s_0)$.

Proposition 2.2. *Let S be a smooth hypersurface in the Riemannian manifold M and let $s_0 \in S$. Let $T(s_0)$ be the tangent manifold of S at s_0. Then, $T(s_0)$ is imbedded hypersurface in M. Furthermore, $T(s_0)$ is the union of all geodesics in M that are tangent to S at s_0 and of length less than I_{s_0}.*

This proposition implies that tangent manifolds are simply geodesic hyperplanes in Euclidean, hyperbolic, and spherical/projective spaces (spaces with codimension one geodesic planes).

Proof. $T(s_0)$ is an imbedded hypersurface since $\exp_{s_0}$ is a diffeomorphism from $\mathcal{D}(0, I_{s_0})$ to $D_0 = D(s_0, I_{s_0})$. By [26, III, 8.3], the geodesics in D_0 through s_0 correspond under $\exp_{s_0}^{-1}$ to segments through 0, and length (less than I_{s_0}) is preserved. So, the geodesics of length less than I_{s_0} tangent to S at s_0 correspond to Euclidean segments of the same length tangent to $\exp_{s_0}^{-1} S$ at 0. □

Definition 2.3. Let M be a Riemannian manifold, S a hypersurface and $s_0 \in S$ with injectivity radius I_{s_0}. Let $r_0 \in (0, I_{s_0})$. Consider the exponential map $\exp_{s_0} : \mathcal{D}(0, I_{s_0}) \to D(s_0, I_{s_0})$. Let $\mathcal{T}$ be the Euclidean tangent plane to $\exp_{s_0}^{-1} S$ in $T_{s_0}M$. The points x and x_m on $S(s_0, r_0)$ are called *mirror points* (or $T(s_0)-$*mirror points*) if and only if x and x_m are images under the exponential map $\exp_{s_0}$ of points on the Euclidean sphere $\mathcal{S}(0, r_0)$ that are mirror-images in $\mathcal{T}$.

Note that the tangent manifold $T(s_0)$ is essentially the image in M of this Euclidean plane $\mathcal{T}$: $T(s_0) = \exp_{s_0}(\mathcal{T} \cap \mathcal{D}(0, I_{s_0}))$. Mirror points are, in some sense, reflections in the tangent manifold $T(s_0)$, and points on $T(s_0)$ are their own mirrors. On constant curvature spaces, mirror points are precisely reflections in the tangent manifolds, which are geodesic hyperplanes.

The definition of mirror points depends on the point s_0 and the tangent manifold $T(s_0)$, but these will be given or they will be clear from context.

Our next theorem is our most general result, and it will be used to prove corollaries of a more geometric nature in Section 3.

Theorem 2.4 (Support Theorem)**.** *Let M be a connected real-analytic Riemannian manifold, and let $S \subset M$ be a real-analytic hypersurface. Let $\mathfrak{A} \subset N$ be open and connected, and let μ be nowhere-zero real-analytic weight on Z. Let $f \in \mathcal{D}'(M)$ and assume that for each $(s, r) \in \mathfrak{A}$ and each $x \in S(s, r) \cap \operatorname{supp} f$, the $T(s)-$mirror point to x is not in* $\operatorname{supp} f$. *Assume* $\operatorname{supp} f$ *is disjoint from the sphere $S(s_0, r_0)$ for some $(s_0, r_0) \in \mathfrak{A}$ and $R_\mu f(s, r) = 0$ for all $(s, r) \in \mathfrak{A}$. Then f is zero on the union of spheres, $\bigcup_{(s,r)\in\mathfrak{A}} S(s, r)$.*

Let $(s, r) \in \mathfrak{A}$. Note that points $x \in S(s, r) \cap T(s)$ are their own $T(s)-$mirrors and so the mirror point assumption in Theorem 2.4 implies that these points are not in $\operatorname{supp} f$.

Under the hypotheses of the theorem, if f is zero near one of the spheres in $\mathfrak{A}$ and integrals of f are zero over all spheres in $\mathfrak{A}$, then f must be zero on all the spheres in $\mathfrak{A}$. One can imagine the set of spheres $S(s, r)$ for $(s, r) \in \mathfrak{A}$ "eating away" at $\operatorname{supp} f$. This is possible because there is an initial sphere $S(s_0, r_0)$ that is disjoint from $\operatorname{supp} f$ and, because of the mirror point assumption, there is no cancellation to make $R_\mu f(s, r)$ zero when f is not zero near $S(s, r)$. The following elementary example demonstrates that such cancellation should be an issue.

Example 2.5. Let S be a hyperplane in $\mathbb{R}^n$. If f is oddly symmetric about S, then $R_1 f \equiv 0$ because the values of f at mirror points cancel when integrated over these spheres. Furthermore, Courant and Hilbert [12, p. 699] showed that the null space of this transform is the set of all functions odd about H. Therefore, if a function f is supported on one side of H, then f is determined by its transform.

3. Applications

In this section, we prove corollaries of our main result, Theorem 2.4, that could suggest data acquisition methods for sonar and thermoacoustic tomography with nonconstant speed of propagation. It should be pointed out that propagation speed can vary as much as ten percent in some commonly scanned objects. In some seismic experiments, ray paths can be rather pathological, too [28]. We start with a definition.

Definition 3.1. Let T be a hypersurface in M and let V be an open connected set. A set $U \subset M$ is *to one side of* T *in* V if and only if $V \setminus T$ consists of two connected open components and $U \cap (V \setminus T)$ is contained in one of these components. We say U is *to one side of* T, if and only if this condition holds with V replaced by M. We say U is *locally to one side of* T *near* $x_0 \in T$ if and only if there exists an open connected neighborhood V of x_0 such that U is to one side of T in V.

Note that U can meet T, but U cannot meet *both* connected components of $V \setminus T$. Often, we will apply this definition to $T = T(s_0)$ and $V = D(s_0, r_0)$. As long as $r_0 \leq I_{s_0}$, then $T(s_0)$ does divide $D(s_0, r_0)$ into two connected open components.

Sonar. In sonar, typically the sound source travels along a surface S, say the surface of the ocean, and the if the speed of sound is constant and the source and detector are at the same point (under the Born approximation), then the data at $y \in S$ at time t is the integral of the reflection, f, echoing back from the sphere of radius $ct/2$ where c is the (constant) speed of sound, or $R_1 f(y, ct/2)$. So, the data taken are integrals over the spherical wavefronts centered at points on S. If, however the speed of sound is not constant, then the resulting metric is not Euclidean, and the spheres are not Euclidean.

Note that if $\forall (s,r) \in \mathfrak{A},\ S(s,r) \cap \operatorname{supp} f$ is to one side of the tangent manifold $T(s)$ in $D(s, I_s)$, then the mirror points assumption of Theorem 2.4 holds. Our next corollary directly generalizes a theorem in [27] for $\mathbb{R}^n$.

Corollary 3.2. *Let M be a connected real-analytic manifold. Let $S \subset M$ be a real-analytic hypersurface. Let $f \in \mathcal{D}'(M)$ and assume for some $s_0 \in S$, supp f is to one side of the tangent manifold to S at s_0 in $D(s_0, I_{s_0})$. Assume further that f is zero near $T(s_0)$.*

Let $r_0 \in (0, I_{s_0})$. If $R_\mu f(s,r) = 0$ for all $s \in S$ and $r \in (0, r_0)$, then, the open disk $D(s_0, r_0)$ is disjoint from supp f.

This theorem is local since S can be an arbitrarily small manifold containing s_0. Furthermore, this corollary is a generalization of a result of Courant and Hilbert given in Example 2.5.

This theorem has general implications for sonar. If S is the surface of the ocean and s_0 is a point on the surface, then objects in the ocean are almost certain to be only on one side of the tangent plane $T(s_0)$ to the surface of the ocean, S. In sonar, if the speed of sound is not constant, then the wavefronts are not Euclidean spheres but spheres for another metric. If that metric were real-analytic and the surface of the ocean were locally real-analytic, then the result of Corollary 3.2 would show that local sonar data for sources near s_0 could determine objects in the ocean. Of course, the applied problem is numerical, and real-analyticity is not meaningful for finite data. However Corollary 3.2 suggests that such local data could determine objects.

Theorem 4.3 gives a characterization of singularities of f that are stably visible from Radon data $R_\mu f$. Such singularities are conormal to the spheres being integrated over. Even though reconstruction is possible by Corollary 3.2, inversion would be highly ill-posed if the set S is small since many singularities would not be stably visible. This is discussed more fully in [27].

Thermoacoustic Tomography. In thermoacoustic tomography the data can be modeled as spherical integrals over spheres centered on a closed surface surrounding the object. In general, it is assumed the spheres are Euclidean, but if the speed of propagation in the object is not constant, then one is in the same situation as sonar with nonconstant speed. One has spherical averages over spheres defined by a non-Euclidean metric (and with centers on a surface), and our results provide uniqueness theorems for this case.

Definition 3.3. Let M be a Riemannian manifold and let $C \subset M$. Then, C is *star-shaped* about $s_0 \in C$ if and only if for each $x \in C$, the shortest geodesic from s_0 to x is in C. C is *convex* if and only if for any two points x and y in C, the shortest geodesic between x and y is in C. In both definitions, if there is more than one shortest geodesic, then at least one has to be in C.

The next corollary is valid for fairly general sets.

Corollary 3.4. *Let M be a connected real-analytic manifold with injectivity radius $I_M > 0$ and let S be a real-analytic hypersurface that bounds a compact set C. Assume the diameter of C is less than the injectivity radius I_M. Let $s_0 \in S$ and assume C is star-shaped about s_0. Let f be a distribution supported in C. Let $\epsilon > 0$, and let $a \in (0, \operatorname{diam}(C))$. Assume $R_\mu f(s,r) = 0$ for all $s \in S$ sufficiently near s_0 and all $r \in (a, \operatorname{diam}(C) + \epsilon)$. Then,* $\operatorname{supp} f \subset \overline{D}(s_0, a)$.

This theorem could have the following implication for TCT. The theorem implies that if C surrounds the object and C is star-shaped about $s_0 \in S = \operatorname{bd} C$, then TCT data for points near s_0 can determine the object inside. As with sonar, inversion from this very local data would be highly ill-posed, as indicated by Theorem 4.3.

Because a convex set (with diameter less than I_M) is star-shaped about each of its boundary points, Theorem 3.4 implies our next corollary.

Corollary 3.5. *Let M be a connected real-analytic manifold with injectivity radius $I_M > 0$ and let S be a real-analytic hypersurface that bounds a geodesically convex compact set C. Assume the diameter of C is less than the injectivity radius I_M. Let f be supported in C. Let $\epsilon > 0$, and let $a \in (0, \operatorname{diam}(C))$. Assume $R_\mu f(s,r) = 0$ for all $s \in S$ and $r \in (a, \operatorname{diam}(C) + \epsilon)$. Then,* $\operatorname{supp} f \subset \bigcap_{s \in S} \overline{D}(s,a)$. *If* $a < \operatorname{diam}(C)/2$, *then* $f = 0$.

This corollary allows one to eat away at part of $\operatorname{supp} f$: if integrals of f over spheres of radius greater than a and smaller than $\operatorname{diam}(C) + \epsilon$ centered on points on S are zero, then $\operatorname{supp} f$ must be at most a units from $S = \operatorname{bd} C$.

Several theorems for the classical transform ($\mu = 1$) on Euclidean spheres in $\mathbb{R}^n$ are worth noting. A strong uniqueness result closely related to this corollary was proved in [16, 36], and inversion formulas are given in [8] for integrals over spheres centered on a hyperplane and inside a disk and more generally over a closed surface in [16].

4. The Microlocal Analysis

The proofs of our theorems involve the real-analytic microlocal analysis of R_μ and a deep theorem of Hörmander, Kawai, and Kashiwara, Theorem 5.2, about real-analytic microlocal singularities at the boundary of $\operatorname{supp} f$. We first introduce the real-analytic wavefront set using the ideas of Bros and Iagnolnitzer (Theorem 9.6.3 [23]).

Definition 4.1. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. Then f is real-analytic at x_0 in direction ξ_0 if and only if there are neighborhoods U of x_0 and V of ξ_0 and constants $C > 0$ and $c > 0$ that make the following localized Fourier transform exponentially decreasing for $x \in U$ and all $\xi \in V$:

$$\left| \int_{y \in \mathbb{R}^n} e^{-iy\cdot\lambda\xi} f(y) e^{-\lambda\|y-x\|^2/2}\, dy \right| \le C e^{-c\lambda}\,; \tag{4.1}$$

(x_0, ξ_0) is in the analytic wavefront set of f, $(x_0, \xi_0) \in \mathrm{WF_A}(f)$ if and only if f is not real-analytic at x_0 in direction ξ_0.

This definition implies $\mathrm{WF_A}(f)$ is a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. If one multiplies f by a C_c^∞ cut off function that is one in a neighborhood of x_0, then the analytic wave front set above x_0 does not change. This follows from the fact that the projection to $\mathbb{R}^n$ of the analytic wave front set is the analytic singular support [23, Theorem 8.4.5]. Furthermore, if $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is identified with $T^*(\mathbb{R}^n)$, then $\mathrm{WF_A}(f)$ can be viewed as a subset of the cotangent bundle. Then, this definition can be shown to be invariant under real-analytic coordinate changes [22, Theorem 3.9]. Therefore, it can be generalized to manifolds using local coordinates and with $\mathrm{WF_A}(f) \subset T^*(M)$.

We have the following microlocal regularity theorem.

Theorem 4.2 (Microlocal Regularlity Theorem)**.** *Let M be a real-analytic Riemannian manifold, and let $S \subset M$ be a real-analytic hypersurface. Let $(s_0, r_0) \in N$ and let $f \in \mathcal{D}'(M)$. Let x and x_m be $T(s_0)-$mirror points on $S(s_0, r_0)$ and assume $x_m \notin \operatorname{supp} f$. Assume $(x, \xi) \in N^*(S(s_0, r_0)) \setminus 0$. Finally, assume $R_\mu f(s, r) = 0$ (or is real-analytic) for all (s, r) in an open neighborhood in N of (s_0, r_0). Then, $(x, \xi) \notin \mathrm{WF_A}(f)$.*

The support restriction on f is important so that wavefront at mirror points do not cancel. As noted in Example 2.5, such cancellation might be an issue. Theorem 4.2 is the direct generalization to manifolds of [4, Theorem 3.3].

Before we prove the theorem, we want to highlight an implication of this microlocal analysis for singularity detection in sonar. The proof of Theorem 4.2 is valid in the C^∞ category, and it allows one to determine C^∞ or Sobolev singularities of f from singularities of $R_\mu f$ using the microlocal correspondence. Here is one version.

Theorem 4.3. *Assume M is a C^∞ Riemannian manifold with injectivity radius $I_M > 0$ and let $S \subset M$ be a smooth hypersurface and $s_0 \in S$ and $r_0 \in (0, I_M)$. Assume* $\operatorname{supp} f$ *is to one side of the tangent manifold $T(s_0)$ in $D(s_0, I_M)$ and disjoint from $T(s_0)$. Assume $R_\mu f$ is **not** C^∞ near $(s_0, r_0) \in N$. Then, $\mathrm{WF}(f) \cap N^*S(s_0, r_0) \neq \emptyset$.*

Let S_0 be the part of $S(s_0, r_0)$ on the same side of $T(s_0)$ as $\operatorname{supp} f$. *Equation* (4.2) *or the local version,* (4.6)*, provide a bijective correspondence between covectors in N^*S_0 in $\mathrm{WF}^s(f)$ and covectors above (s_0, r_0) in $\mathrm{WF}^{s+(n-1)/2}(R_\mu f)$.*

So, if $R_\mu f$ is not smooth near (s_0, r_0), then there is wavefront at the point on $S(s_0, r_0)$ given by the microlocal correspondence (4.2) depending on the wavefront of $R_\mu f$. The only singularities of f visible from data near (s_0, r_0) are those conormal to $S(s_0, r_0)$. Other singularities for f do not produce singularities in the data near (s_0, r_0).

The proof of Theorem 4.3 is as follows. Theorem 4.2 is valid in the C^∞ category and the proof is the C^∞ version of our proof. Then, standard results on how Fourier integral operators map wavefront sets (e.g., [39, Theorem 8.5.4]) provide the correspondence of wavefront sets of f and $R_\mu f$. The Sobolev case follows using the arguments given in [35] for the classical Radon transform.

Proof of Theorem 4.2. We outline how to calculate the Lagrangian manifold of the Fourier integral operator R_μ. Because of the injectivity radius assumption, we can transport the calculation back to $\mathbb{R}^n$ using the exponential map at s_0 and then use the fact that Euclidean spheres centered at the origin in the tangent space correspond to geodesic spheres centered at s_0. We also use a derivative calculation for the geodesic distance from [35, 17].

Because R_μ is a Radon transform, it is a Lagrangian distribution associated to $\Gamma = N^*Z \setminus 0$ where N^*Z is the conormal bundle of Z in $T^*(M \times N)$ [18]. To prove the theorem, we need to show R_μ is a Fourier integral operator and then find its microlocal properties. To do this, we examine the maps in the microlocal diagram for R_μ:

$$\begin{array}{ccc} \Gamma = N^*Z \setminus 0 & \xrightarrow{\rho} & T^*N \setminus 0 \\ \Big\downarrow{\scriptstyle\pi} & & \\ T^*M \setminus 0 & & \end{array} \tag{4.2}$$

where π and ρ are the natural projections.

Here is the outline of the proof. To show R_μ is a well-behaved Fourier integral operator, one must first show π really maps to $T^*M\setminus 0$ and ρ really maps to $T^*N\setminus 0$. Then, to prove Theorem 4.2, one must show the following.

(4.3) Let $(s_0, r_0) \in N$. Then, above $\{(x, s_0, r_0) \,|\, x \in S(s_0, r_0) \setminus T(s_0)\}$, ρ is a two-to-one local diffeomorphism that maps corresponding covectors in Γ that lie above $T(s_0)-$mirror points in $S(s_0, r_0)$ to the same point in $T^*_{(s_0,r_0)}N \setminus 0$.

The corresponding covectors are π^{-1} of corresponding covectors (x, η) and (x_m, η') in $N^*(S(s_0, r_0)) \setminus 0$. Under the assumptions of the proposition, the calculus of analytic elliptic Fourier integral operators [37, 39] implies the microlocal smoothness assertion of the theorem: because $(x_m, \eta') \notin \mathrm{WF_A}(f)$, singularities at x_m do not cancel singularities at x. R_μ will be analytic elliptic because the weight μ is real-analytic and nowhere zero. Therefore, if $R_\mu f = 0$ then and $(x_m, \eta') \notin \mathrm{WF_A}(f)$, then $(x, \eta) \notin \mathrm{WF_A}(f)$. The part of the proof after (4.3) is exactly as the proof of the analogous theorem (Theorem 3.3) in [4].

The proof of (4.3) is done in local coordinates since spheres centered at s_0 are images of Euclidean spheres under the exponential map at s_0. We use geodesic normal coordinates to write the distance on M locally in terms of the Euclidean distance on the tangent space. Let $(s_0, r_0) \in N$ and let $\mathcal{B}$ be an open Euclidean ball centered at $0 \in T_{s_0}M$ of radius $r_1 \in (r_0, I_{s_0})$. Then in geodesic coordinates on $\mathcal{B}$, $\exp = \exp_{s_0} : \mathcal{B} \to M$ is a diffeomorphism onto the open geodesic ball $B \subset M$ centered at s_0 of radius r_1 [26, IV 3.4]. Let $\mathcal{V} \subset \mathcal{B}$ be a neighborhood of zero such that B is a normal neighborhood of each point in $V = \exp \mathcal{V}$. This is possible because $y \mapsto I_y$ is lower semicontinuous. By [26, III 8.3 and IV 3.4], the shortest geodesic in M between each point $y \in V$ and each point $x \in B$ lies in B, and the proof of [26, IV 3.6] shows that the square of the distance function, $d^2(x,y)$, is real-analytic on $B \times V$. If $X \in \mathcal{B}$ and $Y \in \mathcal{V}$, then one can write the distance function in terms of the Euclidean metric on $T_{s_0}M$ as

$$d^2(\exp(X), \exp(Y)) = ||X - Y||^2 + c(X,Y)$$

for some real-analytic function c satisfying

$$c(X,0) = \frac{\partial}{\partial X_i} c(X,0) = \frac{\partial}{\partial Y_j} c(X,0) = \frac{\partial^2 c}{\partial X_i \partial Y_j}(X,0) = 0 \quad \forall\ X \in \mathcal{B},\ i,j \in \{1,\dots,n\} \tag{4.4}$$

(see [35, 17] for a proof of (4.4)).

Let $\mathcal{T}$ be the tangent plane to $\exp^{-1} S$ at $0 = \exp^{-1} s_0$. Using a rotation we can assume $\mathcal{T}$ is the plane $Y_n = 0$. Let $\mathcal{S}'$ be a small neighborhood of $0 \in \mathbb{R}^{n-1}$ on which $Y_n = Y_n(Y_1, \dots, Y_{n-1})$ gives analytic coordinates on $\mathcal{V} \cap \exp^{-1} S$. Because of this rotation, $\frac{\partial Y_n}{\partial Y_j}(0) = 0$ for all $j = 1, \dots, n-1$. So, $(Y', r) \to (\exp(Y', Y_n(Y')), r)$ gives local coordinates on N.

For $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ in $\mathbb{R}^n$, we let $X\mathbf{dX} = X_1 dX_1 + \dots + X_n dX_n$, $Y' = (Y_1, \dots, Y_{n-1})$, and $Y'\mathbf{dY}' = Y_1 dY_1 + \dots + Y_{n-1} dY_{n-1}$.

Our assumptions about $\mathcal{S}'$, $\mathcal{V}$, and $\mathcal{B}$, show that

$$\mathcal{Z}' = \{(X, Y', r) \,\big|\, X \in \mathcal{B},\ Y' \in \mathcal{S}', (\exp(X), \exp(Y', Y_n(Y')), r) \in Z\}$$

maps bijectively to an open subset of Z via the map

$$(X, Y', r) \to (\exp(X), \exp((Y', Y_n(Y')), r).$$

Furthermore, as $\mathcal{Z}'$ is defined by the single equation

$$\|X - (Y', Y_n(Y'))\|^2 + c(X, (Y', Y_n(Y'))) - r^2 = 0,$$

the conormal bundle is

$$\begin{aligned}\Gamma' = N^*\mathcal{Z}' \setminus 0 = \big\{ \big(X, Y', r; \alpha\big[\big(2(X-Y) + \nabla_X c(X,Y)\big)\mathbf{dX} \\ -\big(2(X-Y) + \nabla_Y c(X,Y)\big)A(Y')\mathbf{dY}' - r\mathbf{dr}\big]\big) \\ \big|\, (X, Y', r) \in \mathcal{Z}',\ Y = (Y', Y_n(Y')),\ \alpha \neq 0 \big\}\end{aligned} \tag{4.5}$$

where $A(Y')$ is the derivative matrix of the map $Y' \mapsto (Y', Y_n(Y'))$, $A(Y') = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ \frac{\partial Y_n}{\partial Y_1} & \frac{\partial Y_n}{\partial Y_2} & \dots & \frac{\partial Y_n}{\partial Y_{n-1}} \end{pmatrix}$. Since $\nabla Y_n(0) = 0$, the last row of $A(0)$ consists only of zero entries.

Using (4.5), one can see the projection π in (4.2) maps to $T^*M \setminus 0$ and ρ maps to $T^*N \setminus 0$ above s_0 $(Y = 0)$. It is straightforward to show above s_0 that ρ is a two-to-one map away from the tangent plane $\mathcal{T}$. In fact, covectors above $(X_1, \dots, X_{n-1}, X_n)$ and $(X_1, \dots, X_{n-1}, -X_n)$ map to the same points under the projection to $T^*(\mathcal{S}' \times (0,\infty))$. This proves the first statement in (4.3) since these vectors correspond to mirror points on $S(s_0, r_0)$ by Definition 2.3.

We need to show that ρ is an immersion above an arbitrary point $(x_0, s_0, r_0) \in Z$ with x_0 not on the tangent manifold $T(s_0)$. This says $(X_0)_n \neq 0$ where $X_0 = \exp^{-1} x_0$. Since $\mathcal{Z}'$ is defined by the equation $\|X - Y\|^2 + c(X,Y) = r^2$, the implicit function theorem and (4.4) show we can specify $X_n = X_n(X', Y', r)$ as an analytic function in a neighborhood $\mathcal{V}$ of $((X_0)', 0, r_0)$. Let ρ' be the projection ρ in these local coordinates (see (4.5)),

$$\rho'(X', Y', r, \alpha) = (Y', r; -\alpha\big(2(X-Y) + \nabla_Y c(X,Y)\big)A(Y')\mathbf{dY'} - \alpha r\mathbf{dr}\big) \tag{4.6}$$

$$X = \big(X', X_n(X', Y', r)\big), \quad Y = \big(Y', Y_n(Y')\big)\,. \tag{4.7}$$

We need to check that the derivatives $d\rho'/dX_j$, $d\rho'/dY_j$, $j = 1, \dots, n-1$, $d\rho'/dr$, and $d\rho'/d\alpha$ are independent. Since Y' and r are coordinates in the right-hand side of (4.6) and α is given by the last coordinate, these derivatives are independent at $((X_0)', 0, r_0)$. So, the only derivatives we need to calculate are $d\rho'/dX_j$. This is a straightforward calculation using (4.4) and the fact the last row of $A(0)$ consists only of zero entries, and it will be left to the reader. This shows ρ' is an immersion, and so ρ is an immersion. Now that we have established (4.3), the microlocal arguments that show (4.3) implies Theorem 4.2 are the same as the end of the proof (on p. 68) of [4, Theorem 3.3], and they will not be repeated. See also [17, Sections 3.1, 3.2] and its references. □

5. Proofs of the Support Theorems.

We begin by proving a lemma that provides the key to all the proofs. The proof will involve the Microlocal Regularity Theorem 4.2 and a theorem of Kawai, Kashiwara, and Hörmander, Theorem 5.2 below, to eat away at $\operatorname{supp} f$.

Lemma 5.1. *Let M be a real-analytic Riemannian manifold, and let $S \subset M$ be a real-analytic hypersurface. Let μ be nowhere-zero real-analytic weight on Z. Let $f \in \mathcal{D}'(M)$ and $(s_1, r_1) \in N$. Assume x_1 is a point in $S(s_1, r_1) \cap \operatorname{supp} f$. Assume that $\operatorname{supp} f$ is locally to one side of $S(s_1, r_1)$ near x_1. Further, assume the $T(s_1)-$mirror point to x_1 in $S(s_1, r_1)$ is not in $\operatorname{supp} f$. Then, $R_\mu f$ cannot be zero in any neighborhood of (s_1, r_1).*

Proof. Let $S_1 = S(s_1, r_1)$ and let $(x_1, \xi_1) \in N^*S_1 \setminus 0$. Since $\operatorname{supp} f$ is locally to one side of S_1 near x_1, the following theorem implies $(x_1, \xi_1) \in \mathrm{WF_A}(f)$.

Theorem 5.2 (Theorem 8.5.6 [23], [24])**.** *Let $f \in \mathcal{D}'(M)$. Assume Σ is a C^2 hypersurface and $x_1 \in \operatorname{supp} f \cap \Sigma$. Finally, assume $\operatorname{supp} f$ is to one side of Σ near x_1. If $(x_1, \xi_1) \in N^*(\Sigma) \setminus 0$, then $(x_1, \xi_1) \in \mathrm{WF_A}(f)$.*

Theorem 5.2 is a generalization of the fact that if f zero on one side of Σ and $\operatorname{supp} f$ meets Σ, then f cannot be real-analytic near x_1.

Here is the contradiction that proves the lemma. If $R_\mu f$ were zero near (s_1, r_1) then, since the $T(s_1)-$mirror point in S_1 to x_1 is not in $\operatorname{supp} f$, Theorem 4.2 would imply $(x_1, \xi_1) \notin \mathrm{WF_A}(f)$. However, because $x_1 \in \operatorname{supp} f$, Theorem 5.2 implies $(x_1, \xi_1) \in \mathrm{WF_A}(f)$. □

Proof of Theorem 2.4. We assume S is a real-analytic hypersurface in a connected real-analytic manifold M. We let $\mathfrak{A} \subset N$ be open and connected. We assume $\forall (s, r) \in \mathfrak{A}$ and each $x \in S(s, r) \cap \operatorname{supp} f$, the $T(s)-$mirror point to x in $S(s, r)$ is not in $\operatorname{supp} f$. We assume $\operatorname{supp} f$ is disjoint from $S(s_0, r_0)$ for some $(s_0, r_0) \in \mathfrak{A}$.

Let (s_2, r_2) be an arbitrary point in A and assume $S(s_2, r_2)$ meets $\operatorname{supp} f$. Since $\mathfrak{A}$ is open and connected, there is a continuous path $p : [0, 1] \to \mathfrak{A}$, $p(t) = (s(t), r(t))$, $p(0) = (s_0, r_0)$, and $p(1) = (s_2, r_2)$. For $(s, r) \in \mathfrak{A}$ and $\epsilon \in (0, r)$, define the annulus

$$A(s, r, \epsilon) = \{x \in M \,\big|\, r - \epsilon \le d(s, x) \le r\}\,.$$

Because $\mathfrak{A}$ is open, the image $p([0,1])$ is compact, and $\operatorname{supp} f$ is closed, $\exists \epsilon_0 > 0$ satisfying

(5.1) $\forall t \in [0,1],\ \ 0 < r(t) - \epsilon_0$ and $\forall r \in [r(t)-\epsilon_0, r(t)]$, $(s(t), r) \in \mathfrak{A}$.

(5.2) The annulus $A(s(0), r(0), \epsilon_0)$ is disjoint from $\operatorname{supp} f$.

We use these annuli to eat away at $\operatorname{supp} f$. Let t_1 be the smallest number in $[0,1]$ such that the annulus

$$A_1 = A(s(t_1), r(t_1), \epsilon_0)$$

meets $\operatorname{supp} f$. By (5.2), $t_1 > 0$. Because t_1 is the smallest number for which $A(t)$ meets $\operatorname{supp} f$, $\operatorname{supp} f$ does not meet the interior of the annulus A_1. Let $x_1 \in A_1 \cap \operatorname{supp} f$. Without loss of generality assume $x_1 \in S_1 := S(s(t_1), r(t_1))$. Since the $T(s(t_1))-$mirror-point to x_1 is not in $\operatorname{supp} f$ by assumption and $R_\mu f$ is zero near (s_1, r_1), $x_1 \notin \operatorname{supp} f$ by Lemma 5.1. This contradiction proves the theorem. □

Proof of Corollary 3.2. Let $0 < r_1 < r_2 < r_0$. By compactness of $\operatorname{supp} f \cap \overline{D}(s_0, r_2)$, the tangent manifold $T(s_0)$ is a finite distance from $\operatorname{supp} f \cap \overline{D}(s_0, r_2)$, and $\operatorname{supp} f \cap \overline{D}(s_0, r_2)$ is to one side of $T(s_0)$. So, there is a connected neighborhood $S_0 \subset S$ of s_0 such that for each $s \in S_0$, $\overline{D}(s_0, r_2) \cap \operatorname{supp} f$ is to one side of $T(s)$ and disjoint from $T(s)$. Perhaps by shrinking S_0, we can also assume for $(s, r) \in \mathfrak{A} := S_0 \times (0, r_1)$, $S(s, r) \subset \overline{D}(s_0, r_2)$. Because $(s, r) \in \mathfrak{A}$, $S(s, r) \cap \operatorname{supp} f$ is to one side of $T(s)$ and does not meet $T(s)$, if $x \in S(s_0, r) \cap \operatorname{supp} f$, its $T(s)-$mirror point is not. Furthermore, for sufficiently small r, $S(s_0, r)$ does not meet $\operatorname{supp} f$. So, we can use Theorem 2.4 and this set $\mathfrak{A}$ to show $D(s_0, r_1)$ is disjoint from $\operatorname{supp} f$. Since $r_1 < r_0$ is arbitrary, this shows $D(s_0, r_0)$ is disjoint from $\operatorname{supp} f$. □

Proof of Corollary 3.4. Now, we assume S is the boundary of a compact set C that is star-shaped about the point $s_0 \in S$. By assumption, $\operatorname{supp} f \subset C \subset D(s_0, I_M)$. We will first prove a geometric lemma.

Lemma 5.3. $\operatorname{supp} f$ *is to one side of* $T(s_0)$ *in* $D(s_0, I_M)$ *and* $T(s_0) \cap S = \{s_0\}$.

Proof. We pull the picture back to $T_{s_0}M$. Let $\mathcal{T}$ be the Euclidean tangent plane to $\mathcal{S} = \exp_{s_0}^{-1} S$ in $\mathcal{D}(0, I_M)$, and let $\mathcal{C} = \exp_{s_0}^{-1} C$. Let $\mathcal{B} = \mathcal{D}(0, r)$ where $r \in (\operatorname{diam} C, I_M)$. Let $\mathcal{V} \subset \mathcal{D}$ be a neighborhood of zero such that $B = D(s_0, r)$ is a normal neighborhood of each point in $V = \exp \mathcal{V}$. By [26, III 8.3 and IV 3.4], the shortest geodesic in M between each point $y \in V$ and each point $x \in B$ lies in B. Furthermore, rays from $0 \in \mathcal{B} \subset T_{s_0}M$ correspond to geodesics under the exponential map. Therefore $\mathcal{C}$ is star-shaped about $0 = \exp_{s_0}^{-1}(s_0)$.

Assume two points $X_1 \in \mathcal{S}$ and $X_2 \in \mathcal{S}$ are in different components of $\mathcal{B} \setminus \mathcal{T}$. Then, the ray R_1 from 0 to X_1 and the ray R_2 from 0 to X_2 are both in $\mathcal{C}$ because $\mathcal{C}$ is star-shaped about 0. However, the boundary of $\mathcal{C}$, $\mathcal{S}$, is tangent to $\mathcal{T}$ at 0, so if rays from 0 to points on $\mathcal{C}$ are on one side of $\mathcal{T}$, they cannot be on the other. This contradiction proves that $\mathcal{C}$ is to one side of $\mathcal{T}$ and so C is to one side of $T(s_0)$ in $D(s_0, r)$.

To prove $T(s_0) \cap S = \{s_0\}$ we assume there a point $X_1 \neq 0$ in $\mathcal{T} \cap \mathcal{S}$. Then, the segment between 0 and X_1 is in $\mathcal{S}$ because $\mathcal{C}$ is star-shaped about 0 and on one side of $\mathcal{T}$. But $\mathcal{S}$ cannot be compact and real-analytic **and** contain a segment. Therefore, $\mathcal{S} \cap \mathcal{T} = \{0\}$, and $S \cap T(s_0) = \{s_0\}$. □

Now, we will apply Lemma 5.1. Let r_1 be the smallest radius such that $\operatorname{supp} f \subset \overline{D}(s_0, r_1)$. Assume $r_1 > a$. Then, by Lemma 5.3, $\operatorname{supp} f$ is to one side of $T(s_0)$ in

$\overline{D}(s_0, r_1)$ and there are no self-mirror points on $S(s_0, r_1) \cap T(s_0)$. Therefore, for each point in $\operatorname{supp} f \cap S(s_0, r_1)$, its $T(s_0)-$mirror point is not in $\operatorname{supp} f$. So, by Lemma 5.1, f is zero near the sphere $S(s_0, r_1)$. This contradiction shows $\operatorname{supp} f \subset \overline{D}(s_0, a)$. □

References

[1] Mark Agranovsky, Carlos Berenstein, and Peter Kuchment, *Approximation by spherical waves in L^p spaces*, J. Geom. Analysis **6** (1996), 365–383.

[2] Mark Agranovsky and Eric Todd Quinto, *Injectivity sets for a Radon transform and complete systems of radial functions, an announcement*, International Mathematical Research Notes (Duke J.) **11** (1994), 467–473.

[3] ______, *Injectivity sets for Radon transform over circles and complete systems of radial functions*, J. Functional Anal. **139** (1996), 383–414.

[4] ______, *Geometry of Stationary Sets for the Wave Equation in $\mathbb{R}^n$, The Case of Finitely Supported Initial Data*, Duke Math. J. **107** (2001), 57–84.

[5] ______, *Stationary Sets for the Wave Equation in crystallographic domains*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2439–2451.

[6] Gaik Ambartsoumian and Peter Kuchment, *On the injectivity of the circular Radon transform arising in thermoacoustic tomography*, Inverse Problems **21** (2005), 473–485.

[7] ______, *A range description for the planar circular Radon transform*, Tech. report, Texas A&M University, 2005.

[8] L-E. Andersson, *On the determination of a function from spherical averages*, SIAM J. Math. Anal. **19** (1988), 214–232.

[9] Aleksei Beltukov, *Sonar transforms*, Ph.D. thesis, Tufts University, 2004.

[10] Carlos A. Berenstein and Lawrence Zalcman, *Pompeiu's problem on symmetric spaces*, Comment. Math. Helvetici **55** (1980), 593–621.

[11] Margaret Cheney, *Tomography problems arising in synthetic aperture radar*, Contemporary Mathematics **278** (2001), 15–28.

[12] Richard Courant and David Hilbert, *Methods of Mathematical Physics*, vol. II, Wiley-Interscience, New York, 1962.

[13] Alexander Denisjuk, *Integral geometry on the family of semi-spheres*, Fractional Calculus and Applied Analysis **2** (1999), 31–46.

[14] ______, *On numerical reconstruction of a function by its arc means with incomplete data*, Tech. report, Brest State University, Belarus, 1999.

[15] J. Fawcett, *Inversion of N-dimensional spherical averages*, SIAM J. Appl. Math. **65** (1985), 336–341.

[16] David Finch, Sarah Patch, and Rakesh, *Determining a function from its mean values over a family of spheres*, SIAM J. Math. Anal. **35** (2004), 1213–1240.

[17] Eric Liviu Grinberg and Eric Todd Quinto, *Morera Theorems for Complex Manifolds*, J. Functional Analysis **178** (2000), 1–22.

[18] Victor Guillemin and Shlomo Sternberg, *Geometric Asymptotics*, American Mathematical Society, Providence, RI, 1977.

[19] Sigurdur Helgason, *The totally-geodesic Radon Transform on constant curvature spaces*, Contemporary Mathematics **113** (1990), 141–149.

[20] ______, *Geometric analysis on symmetric spaces*, American Mathematical Society, Providence, RI, 1994.

[21] ______, *The Radon Transform, Second Edition*, Birkhäuser, Boston, 1999.

[22] Lars Hörmander, *Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential equations with Analytic Coefficients*, Comm. Pure Appl. Math. **24** (1971), 671–704.

[23] ______, *The Analysis of Linear Partial Differential Operators*, vol. I, Springer Verlag, New York, 1983.

[24] Akira Kaneko, *Introduction to Hyperfunctions*, Kluwer, New York, 1989.

[25] J. Klein, *Rekonstruktionsverfahren für SAR: Inversion sphärischer Durchschnitte*, Master's thesis, Westfälische Wilhelms - Universität, Institut für Numerische und instrumentelle Mathematik, Münster, Germany, 2003.

[26] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of Differential Geometry*, Classics Library, vol. 1, John Wiley and Sons, New York, 1996.

[27] Alfred K. Louis and Eric Todd Quinto, *Local Tomographic Methods in SONAR*, Surveys on Solution Methods for Inverse Problems (Vienna/New York) (D.Colton, H.Engl, A.Louis, J.McLaughlin, and W. Rundell, eds.), Springer Verlag, 2000.

[28] A.E. Malcolm, M.V. de Hoop, and J.H. La Rousseau, *The applicability of dip moveout/azimuth moveout in the presence of caustics*, Geophysics 70(2005), S1-S17.

[29] M.M. Nessibi, L.T. Rachdi, and K. Trimeche, *Ranges and inversion formulas for spherical mean operator and its dual*, J. Math. Anal. Appl. **196** (1995), no. 3, 861–884.

[30] S.J. Norton, *Reconstruction of a reflectivity field from line integrals over circular paths*, J. Acoust. Soc. Am. **67** (1980), 853–863.

[31] ———, *Reconstruction of a two-dimensional reflecting medium over a circular domain: exact solution*, J. Acoust. Soc. Am. **67** (1980), 1266–1273.

[32] S.J. Norton and Melvin Linzer, *Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures*, IEEE Trans. Biomedical Engineering **28** (1981), 200–202.

[33] Victor Palamodov, *Reconstruction from Limited Data of Arc Means*, J. Fourier Analysis and Applications **6** (2000), 26–42.

[34] ———, *Reconstructive Integral Geometry*, Monographs in Mathematics, Birkhauser, Boston, MA, USA, 2004.

[35] Eric Todd Quinto, *Pompeiu transforms on geodesic spheres in real analytic manifolds*, Israel J. Math. **84** (1993), 353–363.

[36] A.G. Ramm, *Injectivity of the spherical mean operator*, Comptes Rendus de l'Académie des sciences, Série I, Mathematique **335** (2002), 1033–1038.

[37] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudodifferential equations*, Lecture Notes in Mathematics, vol. 287, Springer Verlag, New York, 1973.

[38] Thomas Schuster and Eric Todd Quinto, *On a regularization scheme for linear operators in distribution spaces with an application to the spherical Radon transform*, SIAM J. Appl. Math. **65** (2005), no. 4, 1369–1387.

[39] F. Treves, *Introduction to Pseudodifferential and Fourier integral operators, Volumes 1 & 2*, Plenum Press, New York, 1980.

[40] V. V. Volchkov, *Theorems on the inversion of the local Pompeiu transform on a quaternion hyperbolic space*, Algebrai Analiz Nauk **15** (2003), 753–771.

[41] Y. Xu, L. Wang, G. Ambartsoumian, and P. Kuchment, *Reconstructions in limited view thermoacoustic tomography*, Medical Physics **31** (2004), no. 4, 724–733.

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