Local Tomographic Methods in SONAR

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Abstract. Tomographic methods are described that will reconstruct object boundaries in shallow water using SONAR data. The basic ideas involve microlocal analysis, and they are valid under weak assumptions even if the data do not correspond exactly to our model.

1 Introduction

Integrals over spheres are important in pure mathematics [12], [20], [22] and in applications in partial differential equations [15] and for physical problems including SONAR [10] [21], seismic testing [21], and RADAR [4]. In this article, we will describe the application to SONAR and geophysical testing and prove a general uniqueness theorem for local data. We will give a singularity detection method for the linear problem that requires only local data. We will explain why this method is valid for data that do not fit our model as long as certain fairly weak assumptions hold. Our results are all valid in any dimension, in particular, $n = 2$ and $n = 3$.

In each of these applied problems, after a linearization, the original inverse problem is reduced to an inverse problem for spherical integrals over spheres with restricted centers. Let A be a hypersurface in $\mathbb{R}^n$ and let $a \in A$. Let $r > 0$. Then, the sphere centered at a and of radius r is defined

$$S(a,r) = \{x \in \mathbb{R}^n \,\big|\, |x-a| = r\}\,. \tag{1}$$

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Now, let f be a continuous function, $f \in C(\mathbb{R}^n)$. We define the spherical average of f over $S(a,r)$ to be

$$Rf(a,r) = \int_{x \in S(a,r)} f(x) dA(x) \tag{2}$$

where dA is the area measure on this sphere.

In seismology or SONAR the acoustic wave equation is

$$n^2(x)u_{tt} = \Delta u + \delta(t)\delta(x - a_0) \text{ where } \ a_0 \in A$$

and A is a small section of the surface of the earth. After linearization, the determination of $n^2(x)$ from back-scattered data is equivalent to inversion of $R(n^2)(a,r)$, with centers on A [16], [21]. Knowing n^2 or at least the discontinuities of n^2 tells boundaries of objects in the water.

This linearized model is reasonable from a practical standpoint when the speed of sound in the ambient water is fairly constant. This would occur in water of depth less than one hundred feet with fairly constant temperature (private communication, R. Barakat). Since the speed of sound is constant in shallow water with constant temperature, a pulse travels from a point source, a, making a spherical wavefront. The sound that is reflected back to the source at time t gives the amount reflected back from the sphere centered at a and radius $t/2$ times the speed of sound (assuming no multiple reflections). See also [14] for practical information about SONAR.

Another inverse scattering problem is to find the scatterer, $q(x)$

$$u_{tt} = \Delta u + q(x)u + \delta(t)\delta(x - a_0) \text{ where } a_0 \in A$$

and A is a small section of the surface of the earth. After linearization, the determination of $q(x)$ from the response at a_0 is equivalent to inversion of the spherical transform R.

A two-dimensional linearized travel-time problem which reduces to integrals over circles with centers on a curve is discussed in [10], [16].

In each of these problems, one wants to find a function or distribution f from integrals over spheres (or circles) with centers on a given surface A (or curve in the plane). In the case of SONAR or geophysical testing, A is some part of the surface of the earth. In these practical problems, the distribution f is assumed to be zero on one side of the surface (its support, $\operatorname{supp} f$, is on the other side of this surface).

Much is known in the case when the surface $A = \mathcal{P}$ is a hyperplane in $\mathbb{R}^n$ and $Rf(a,r)$ is known for all $a \in \mathcal{P}$ and all $r > 0$. If f is odd about the hyperplane $\mathcal{P}$, then all spherical integrals over spheres centered on $\mathcal{P}$ are zero by symmetry. Courant and Hilbert [8] proved that any continuous *even* function is uniquely determined by its spherical integrals for spheres with centers on a hyperplane. Thus, the null space of this transform is the set of all odd functions. Therefore, any function supported on one side of $\mathcal{P}$ is uniquely determined by spherical integrals.

Inversion formulas are given for the spherical transform over spheres centered on A when A is a circle in the plane [17], when A is a plane in $\mathbb{R}^3$ [10], and when A is a hyperplane in $\mathbb{R}^n$ [4]. The formulas in [10] and [4] involve back projection, a dual operator to R, composed with a non-local Fourier integral operator. Palamodov [18] and Denisjuk [9] developed mappings which reduce this problem to inversion of the classical Radon transform. Their inversion method is local for odd dimensions (as would be expected from a dimension count). We will discuss this approach a little more in §4. These inversion methods require data $Rf(a,r)$ for spheres of arbitrary large radius to recover the value of $f(x)$ because the back projection requires this.

Very little is known if the set of centers, A, is not a hyperplane or circle. For the problem of integration over circles in the plane, the main theorem of [2] shows that, if f is compactly supported, then f is determined by integrals over circles with centers on all curves A except Coxeter systems of lines (lines intersecting at one point with equally spaced angles). This says that if A is any curve in the plane that is not a line segment, then inversion of $Rf(a,r)$ with centers $a \in A$ is possible. Partial results exist in $\mathbb{R}^n$ (e.g., [3]). It is shown in [1] that if A is the boundary of a compact smooth set in $\mathbb{R}^n$, then f is determined by spherical integrals over spheres with centers on A if f decreases sufficiently rapidly at infinity.

Much work has been done on other inverse scattering problems including models using double integrals over spheres [5] and inversion methods with error estimates for scattering with one direction of incidence and all directions of scatter [7].

This article is organized as follows. In §2, we will develop the basic ideas for understanding singularity detection. In §3, we will describe how R detects singularities, and we will also prove new uniqueness and support theorems for this transform with local data. Finally in §4, we will discuss practical aspects of the problem including numerical implementations and limitations of the model as well as cases in which the model is not satisfied, but the method will still find singularities.

2 The Mathematical Preliminaries

In this section, we talk about singularities using the ideas of Fourier transforms, Sobolev spaces, and wavefront sets. For $f \in L^1(\mathbb{R}^n)$ the Fourier transform and its inverse evaluated on f are

$$\begin{aligned} \mathcal{F}f(y) &= \int_{x\in\mathbb{R}^n} f(x)e^{-ix\cdot y}dx \\ \mathcal{F}^{-1}f(x) &= \frac{1}{(2\pi)^n}\int_{y\in\mathbb{R}^n} f(y)e^{ix\cdot y}dy \end{aligned} \tag{3}$$

Sobolev spaces are generalizations of L^2 spaces that categorize which derivatives of a function are in L^2. The Sobolev space $H^s(\mathbb{R}^n)$ is defined for $s \in \mathbb{R}$ as the set of all distributions f for which the Fourier transform $\mathcal{F}f$ is a function that satisfies

$$\|f\|_s^2 = \int_{y\in\mathbb{R}^n} |\mathcal{F}f(y)|^2(1+|y|^2)^s dy < \infty\,. \tag{4}$$

We can use these ideas and localize in the Fourier domain to get more precise information about singularities, the wavefront set.

Definition 1. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $x_0 \in \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n \setminus 0$. Then, f is smooth microlocally near (x_0, ξ_0) if and only if there is a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(x_0) \neq 0$ and there is an open cone V containing ξ_0 such that $\mathcal{F}(\varphi f)(y)$ is rapidly decreasing in V. If f is* **not** *smooth microlocally near (x_0, ξ_0), then we say $(x_0, \xi_0) \in \mathrm{WF}(f)$.*

One can define Sobolev wavefront set, which captures more precise information about singularities: singularities that are not in H^s microlocally [19].

Definition 2. *Let f be a distribution and let $x_0 \in \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n \setminus 0$. Let $s \in \mathbb{R}$. Then, f is microlocally in H^s near (x_0, ξ_0) if and only if there is a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(x_0) \neq 0$, and there is an open cone V containing ξ_0 such that $\int_{\xi\in V} |\mathcal{F}(\varphi f)(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$. If f is* **not** *microlocally H^s near (x_0, ξ_0), then we say $(x_0, \xi_0) \in \mathrm{WF}^s(f)$.*

If $(x_0, \xi_0) \notin \mathrm{WF}(f)$, then for any s, f is microlocally H^s near (x_0, ξ_0). It can be shown using this definition (and a compactness argument on S^{n-1}) that if f is in H^s in every direction at every point in $\mathbb{R}^n$, then f is in $H^s(\mathbb{R}^n)$.

These definitions generalize to manifolds by having $(x; \xi)$ live on the cotangent space of the manifold. We will consider only the manifolds $\mathbb{R}^n$ and $A \times (0, \infty)$, so we will use the standard basis of $T^*\mathbb{R}^n$: $\{\mathbf{dx}_j \,|\, j = 1, \dots, n\}$ where $\mathbf{dx}_j$ is the dual covector to $\partial/\partial x_j$. For $x \in \mathbb{R}^n$, this gives global coordinates on $T_x^*\mathbb{R}^n$. Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, then we define

$$w \cdot \mathbf{dx} = \sum_{j=1}^n w_j \mathbf{dx}_j\,.$$

So, if $\xi_0 \in \mathbb{R}^n$ is the vector in Definitions 1 and 2, then $(x_0, \xi_0 \cdot \mathbf{dx})$ is the corresponding covector in the wavefront set.

Let A be a hypersurface. We get covectors on T^*A as follows. Let $a \in A$ and let T_a be the hyperplane in $\mathbb{R}^n$ tangent to A at a. Then, for $w \in T_a - a$, the translate of T_a to the origin,

$$w \cdot \mathbf{dx} \in T_a^* A\,.$$

So, a covector in $T^*(A \times (0, \infty))$ is of the form $(a, r; w \cdot \mathbf{dx} + s\mathbf{dr})$ where $w \in T_a - a$ and $s \in \mathbb{R}$.

3 Mathematical Results

First, we give a precise description of how the spherical transform detects singularities, then we prove local uniqueness theorems.

Theorem 3 (Microlocal regularity of R). *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let A be a smooth hypersurface. Let $a_0 \in A$ and let T_{a_0} be the hyperplane tangent to A at a_0. Assume* $\operatorname{supp} f$ *lies on one side of T_{a_0}. Let $\alpha \neq 0$ and $r_0 > 0$ and let $x_0 \in S(a_0, r_0)$ and let ξ_0 be normal to $S(a_0, r_0)$ at x_0. Let $\xi_0 = (x_0 - a_0) \cdot \mathbf{dx}$ and let $\eta_0 = -(P_a(x_0 - a_0) \cdot \mathbf{dx} + r_0 \mathbf{dr})$ where P_a is the orthogonal projection onto the hyperplane $T_a - a$. Then,*

$$(x_0; \alpha\xi_0) \in \mathrm{WF}(f) \text{ if and only if } (a_0, r_0; \alpha\eta_0) \in \mathrm{WF}(Rf) \tag{5}$$

Furthermore,

$$(x_0; \alpha\xi_0) \in \mathrm{WF}^s(f) \text{ if and only if } (a_0, r_0; \alpha\eta_0) \in \mathrm{WF}^{s+(n-1)/2}(Rf) \tag{6}$$

The covector $\xi_0 = (x_0 - a_0) \cdot \mathbf{dx}$ is conormal to the sphere $S(a_0, r_0)$ at x_0 (it corresponds to a vector normal to this sphere at x_0) so the theorem gives information about singularities of f conormal to $S(a_0, r_0)$. If Rf is smooth (or in $H^{s+(n-1)/2}$) in the direction η_0 in the theorem, then f is smooth (or in H^s) in direction ξ_0. So, smoothness of the spherical transform of f corresponds to smoothness of f in directions conormal to $S(a_0, r_0)$. More precisely, let $\mathcal{A} \subset A \times (0, \infty)$ be the open subset over which data are taken, then

$$\mathrm{WF}^s(f) \bigcap \Big(\bigcup_{(a,r) \in \mathcal{A}} N^*(S(a,r)) \Big) \tag{7}$$

is the set of H^s-stably reconstructed wavefront directions.

This is true because R satisfies (6) for data satisfying the condition of Theorem 3. Directions $(x_0, a\xi_0)$ satisfying (6) are the ones in the union in (7). These are directions conormal to the spheres $S(a, r)$ in the data set (for $(a, r) \in \mathcal{A}$).

This theorem says nothing about "invisible" singularities (ones not in (7)) but one can easily come up with functions f with singularities in directions not in (7) such that Rf is smooth; these singularities of f disappear in Rf.

This can be used to understand which boundaries of f (boundaries of objects in the ocean) are detectable from local sonar data. Let A be a smooth open set on the surface of the ocean. Let the reflector f lie below T_a for all $a \in A$. Let SONAR data be given on an open connected set $\mathcal{A} \subset A \times (0, \infty)$. Then, singularities of f conormal to $S(a, r)$ will be detectable from the given data for all $(a, r) \in \mathcal{A}$. But, singularities not conormal the sphere will not be stably detected by data near (a, r). For example, if $A = \mathcal{P}$ is a horizontal plane, then vertical boundaries will not be stably detected by any data with centers on $\mathcal{P}$ because no sphere centered on $\mathcal{P}$ has vertical conormals below the surface, $\mathcal{P}$.

Furthermore, according to (7), if $\mathcal{A} = A \times (0, R)$ for some $R > 0$, then more wavefront directions are stably visible near A than far away because the union in (7) includes more directions for points near A than far from A.

Note that Theorem 3 says nothing about points $x_0 \in S(a_0, r_0)$ that are on the equator $S(a_0, r_0) \cap T_{a_0}$. In fact, Theorem 3.3 of [3] makes no conclusion about such points.

This proof is related to Theorem 3.3 of [3], and it will be given in a future article. In particular, (5) and (6) follow from the fact that, for distributions f supported on one side of T_{a_0}, R is an elliptic Fourier integral operator that satisfies the Bolker Assumption [11].

Palamodov [18] has done a careful analysis of singularities of this operator in the plane when $A = \mathcal{P}$ is a line. He has L^2 estimates even for the invisible directions (ones not conormal to spheres in the data set). This special structure lends itself to more precise information.

Our next theorem is a very general local uniqueness theorem.

Theorem 4 (Local Uniqueness for the Spherical Transform). *Let A be a real analytic hypersurface in $\mathbb{R}^n$ and let $\mathcal{A} \subset A \times (0, \infty)$ be open and connected. Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Assume for all $(a, r) \in \mathcal{A}$, that f is supported on one side of T_a, the hyperplane tangent to A at a. Assume for some $(a_0, r_0) \in \mathcal{A}$ that $S(a_0, r_0)$ is disjoint from* supp f. *Then,*

$$f = 0 \quad on \quad \bigcup_{(a,r)\in\mathcal{A}} S(a, r)\,.$$

In this theorem, we must assume A is real-analytic because there are counterexamples to uniqueness for C^∞ Radon transforms. Local uniqueness theorems are known if $A = \mathcal{P}$ is a hyperplane. In [8] it is shown that if U is an open subset of a plane $\mathcal{P}$, and f is zero on one side of $\mathcal{P}$ and $Rf(a, r) = 0$, for all $(a, r) \in \mathcal{A} = U \times (0, \infty)$ then $f \equiv 0$. In [4] uniqueness is shown if $\mathcal{A}$ is the set of all spheres centered on $\mathcal{P}$ and lying inside a given sphere $S(a_0, r_0)$. In this case, $f = 0$ inside $S(a_0, r_0)$. Theorem 4 is stronger than the ones in [8] and [4] since A is not restricted to be a plane and the sets of spheres is more general.

Here is how one could use this theorem as a guide in choosing which SONAR data to use in exploration. Let A be a small open connected set on the surface of the ocean. Assume A and the reflector in the ocean, f, satisfy the conditions of Theorem 4. Assume data are given on A for all spheres of radius less than some r_0. So, $\mathcal{A} = A \times (0, r_0)$. Then, f is determined on $\cup\{S(a, r) \,|\, (a, r) \in \mathcal{A}\}$ by SONAR data on spheres in $\mathcal{A}$. Furthermore, the set of wavefront directions in (7) are stably reconstructed.

The proof of Theorem 4 is similar in spirit to the proof in [6] and it will be given in a future article.

4 Discussion and Future Directions

There is some debate whether the Born approximation and spherical integrals are the right model for the SONAR problem when sources and detectors are at the same location. However, even if the model is inaccurate, as long as a reasonable assumption about singularities is valid, the analysis would still be valid.

In particular, as long as singularities of the objects conormal to the spherical wavefronts result in singularities of the data (as described by (5) and (6)) then backprojection singularity detection algorithms would work. If real SONAR data of a scatterer f has the same singularities as Rf would have, then R^* of the data would reproduce the visible singularities of f. This is because the backprojection operator takes singularities of Rf (and so singularities of anything with the same singularities as Rf) to the visible singularities of f.

This analysis suggests that one consider local singularity detection methods. When $A = \mathcal{P}$ is a plane (or line in $\mathbb{R}^2$), Palamodov [18] and Denisjuk [9] have developed an inversion method for sonar data that reduces the problem to inversion of the classical Radon transform for functions supported in the unit disk, D. In order to get data over all lines in D, one needs sonar data over all spheres. They have proposed using limited angle inversion methods on this Radon data. One of the authors and, independently Peter Kuchment, have suggested using local Lambda tomography on this data. A student of the second author, Alexander Beltukov, is working on implementing this idea. One of the authors has proposed using a sort of local CT directly on the SONAR data. Let R^* be a backprojection operator ($R^*g(x)$ is the average of $g(a,r)$ over all $(a,r) \in \mathcal{A}$ with $x \in S(a,r)$ in a smooth weight that is zero near the boundary of $\mathcal{A}$). Then, the singularities of ΔR^*Rf will give the visible singularities of f, at least theoretically. Mr. Beltukov will investigate these methods, too.

One advantage of using local methods on the sonar data directly (as opposed to mapping to the classical Radon transform), is that one does not have to assume the surface of the ocean is planar; it can have waves. These methods will be presented in a future article.

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