Contemporary Mathematics

Remarks on stationary sets for the wave equation

Mark L. Agranovsky and Eric Todd Quinto

Abstract. Stationary sets are sets of points in $\mathbb{R}^n$ where the solution to the wave equation (with zero initial position) is zero for all time. Stationary sets, $S[f]$, depend on the initial velocity, $f(x)$. We also consider sets of points, $S_T[f]$, where the solution vanishes starting from a certain moment in time $T(x)$ that depends on the point $x \in \mathbb{R}^n$. For the wave equation in even dimensional space, $\mathbb{R}^{2n}$, and any function $T : \mathbb{R}^n \to [0, \infty)$, we prove that $S_T[f] = S[f]$ when the initial velocity f has compact support, and we prove some support restrictions on f in odd dimensions.

We also study stationary sets in the case when the outer boundary of $\operatorname{supp} f$ is convex and prove facts that support a conjecture raised earlier by the authors which states that stationary hypersurfaces are cones.

1. Introduction

Let us consider the following initial value problem for the wave equation in $\mathbb{R}^n$:

$$\begin{aligned} \Delta u = u_{tt} \quad u = u(x,t),\ x \in \mathbb{R}^n,\ t > 0,\\ u(x,0) = 0, \quad x \in \mathbb{R}^n,\\ u_t(x,0) = f(x),\ x \in \mathbb{R}^n\ , \end{aligned} \tag{1.1}$$

where f is in the space of continuous functions of compact support, $f \in C_c(\mathbb{R}^n)$.

Let $T : \mathbb{R}^n \to [0, \infty)$, and define the $T-$stationary set $S_T[f]$ as

$$S_T[f] = \{x \in \mathbb{R}^n : u(x,t) = 0, \quad \forall\ t > T(x)\}\,. \tag{1.2}$$

We define $S[f] = S_0[f]$. Thus, $S_T[f]$ is the set of all points in $x \in \mathbb{R}^n$ where the solution $u(x,t)$ to (1.1) vanishes for all time $t > T(x)$ assuming u is identically zero at $t = 0$ but with initial velocity given by $f(x)$.

This problem has a rich history, in part due to its relation to nodal sets for eigenfunctions for the Laplacian [**8, 9, 11, 12**]. One can show $S[f]$ is the intersection of nodal sets, (e.g., [**5**]). Lin and Pinkus originally considered this problem in relation to approximation theory [**16**], and there are many results on the structure of $S[f]$. If f does not grow too large at infinity, Agranovsky, Berenstein, and Kuchment [**2**] showed that $S[f]$ contains no closed bounded surface provided f decays

2000 *Mathematics Subject Classification.* Primary: 35L05, 44A12 Secondary: 35B05, 35S30.

The first author was supported by the Israel Science Foundation (grant No.279/02-1).

The second author was partially supported by Tufts University FRAC and the National Science Foundation under grants DMS-0200788 and DMS-0456858.

at infinity sufficiently rapidly, and they found the critical rate of decay. If f has compact support in $\mathbb{R}^2$, then Agranovsky and Quinto [**3, 4**] completely characterized the structure of $S[f]$. This work uses heavily the structure of zero sets of harmonic polynomials and microlocally analysis. Using this as a base, the authors conjectured the structure of $S[f]$ in higher dimensions and f of compact support (Conjecture 2.1). Recently, Finch, Patch, and Rakesh, discovered an elegant inversion formula for the spherical transform. Ambartsoumian and Kuchment [**1**] and Boman (unpublished) have used these ideas and PDE methods to begin to prove this conjecture in higher dimensions in an elegant way.

Now introduce the spherical transform

$$Rf(x,r) = \int_{S(x,r)} f(y)dA(y), \tag{1.3}$$

where $S(x,r) = \{y \in \mathbb{R}^n \,\big|\, |y-x| = r\}$ and $dA(y)$ is the normalized area measure on the sphere $S(x,r)$. Let $T : \mathbb{R}^n \to [0,\infty)$, then we define

$$N_T[f] = \{x \in \mathbb{R}^n \,\big|\, Rf(x,r) = 0,\ \forall r > T(x)\} \tag{1.4}$$

and $N[f] = N_0[f]$.

It is clear that $S[f] \subset S_T[f]$ and $N[f] \subset N_T[f]$, but the relationship between these sets is stronger. In [**3**] we proved $S[f] = N[f]$.

THEOREM 1.1. *Let $f \in C_c(\mathbb{R}^n)$ and $T : \mathbb{R}^n \to [0,\infty)$ be arbitrary. If n is even, then $S_T[f] = N[f] = S[f]$. However, $S_T[f]$ can be a proper subset of $N_T[f]$.*
If n is odd, then $S_T[f] = N_T[f] \subset S[f] = N[f]$.

This theorem reflects the fact that Huygens' Principle is strict in odd dimensions but not in even dimensions. That is, in odd dimensions, the value of the solution $u(x_0,t_0)$ depends on the initial data only on the sphere centered at x_0 of radius t_0 (see (4.5)). This suggests why $S_T[f] = N_T[f]$ in odd dimensions; both sets depend only on the integrals of f on individual spheres. In even dimensions, $u(x_0,t_0)$ depends on the initial data on the entire disk centered at x_0 of radius t_0 (see (4.2) which cannot be reduced to an integral over $S(x_0,t_0)$ because of the fractional power $(n-3)/2$ when n is even). So, in even dimensions, $S_T[f]$ does not have to be equal to $N_T[f]$, and this can be shown by simple examples. This also suggests why $S_T[f] = N[f]$ for n even: points in both sets are determined by zero integrals over all concentric spheres.

In [**3, 4**], we characterized sets $S[f]$ for $f \in C_c(\mathbb{R}^2)$ in terms of Coxeter systems of lines. A Coxeter system of lines is a rigid motion of the set of lines through N^{th} roots of unity for some $N \in \mathbb{N}$. Our characterization and Theorem 1.1 lead to the following corollary.

COROLLARY 1.2. *Let $T : \mathbb{R}^2 \to [0,\infty)$ and $f \in C_c(\mathbb{R}^2)$. Then $S_T[f]$ is either empty, a finite set, or the union of a finite set and a Coxeter system of lines.*

In odd dimensions, $S_T[f]$ can be larger than $S[f]$ but there can still be support restrictions on f. Here is one such restriction when $T = T_0$ is a constant.

THEOREM 1.3. *Let n be odd, let $T_0 > 0$ be constant, and let $f \in C_c(\mathbb{R}^n)$. Assume $S_{T_0}[f]$ contains an open set U. Then* $\operatorname{supp} f$ *is contained inside each sphere $S(x,T_0)$ for each $x \in U$.*

A more precise support restriction will be given in Section 4 along with the proofs of Theorems 1.1 and 1.3.

2. $S[f]$ when the outer boundary of $\operatorname{supp} f$ is convex

Describing stationary sets $S[f]$ in dimension $n > 2$ for $f \in C_c(\mathbb{R}^n)$ is still open. In [**3**] we formulated the following.

CONJECTURE 2.1 ([**3**]). *Assume $f \in C_c(\mathbb{R}^n)$ is not identically zero. Then the stationary set $S[f]$ has the form $V \cup (a + h^{-1}(0))$, where V is an algebraic variety of codimension greater than 1, $a \in \mathbb{R}^n$ and h is a nonzero homogeneous polynomial. Moreover, h divides a nonzero homogeneous harmonic polynomial (spherical harmonic).*

The difficult part in proving the conjecture is to check that the $(n-1)-$dimensional part of $S[f]$ is a cone with respect to some vertex a. It is not even clear how to distinguish a potential vertex. It is known that for compactly supported f the set $S[f]$ is algebraic (see [**16, 3**]) and once one knows that the hypersurface part of $S[f]$ is a cone, then it is automatically defined by zeros of a divisor of a nonzero homogeneous harmonic polynomial [**3, 7**].

We now consider a particular case of the problem when the support of f is bounded by a convex closed surface.

DEFINITION 2.2. We say a point a in a hypersurface $S \subset \mathbb{R}^n$ is *regular* if and only if there is a C^2 hypersurface A such that $a \in A$ and $A \subset S$.

The *outer boundary* of B is the part of $\operatorname{bd} B$ that meets the unbounded component of $\mathbb{R}^n \setminus B$. The compact set B is *bounded by E* if and only if E is the outer boundary of B.

Let E be a convex set and let ℓ be a line. We say ℓ is *orthogonal to E at $a \in E$* **iff** E is to one side of the hyperplane through a and perpendicular to ℓ.

For example, the point $(0, 0)$ is a regular point of the curve $xy = 0$. Note that multiple lines can be orthogonal to the set E at points at which E is not regular.

THEOREM 2.3. *Assume $f \in C_c(\mathbb{R}^n)$ and $\operatorname{supp} f$ is bounded by the convex surface E. Then $S[f]$ is a ruled surface at every point of $S[f]$ outside E. Each ruling line is entirely contained in $S[f]$ and meets the surface E orthogonally at its two points of intersection.*

This result was obtained by the authors several years ago but was never published, Recently, Ambartsoumian and Kuchment [**1**] obtained the same result if the outer boundary of $\operatorname{supp} f$ is convex or if it is C^2, using a technique different from that we used. Their proof is purely within PDE methods and is motivated by the elegant inversion formula and ideas in [**13**]. It is felt that their approach, [**1**], may lead to even stronger results. Independently, Jan Boman discovered the same underlying principle.

Because the basic support theorem we use generalizes to manifolds [**17**], this theorem should also generalize, and we are working on this. The limitation is that the manifold must satisfy some sort of Poisson-Kirchhoff formula (which is true in some manifolds including hyperbolic spaces). Probably a theorem like Theorem 1.1 would hold when Huygens' Principle holds on the manifold, but this is future work.

Our method is based on analysis of analytic wavefront sets. The idea behind it is the following. The symmetry principle for the wave equation says that if a solution $u(x, t)$ vanishes on a hyperplane S for all $t > 0$, then $u(x, t)$ and, in particular, the initial data $f(x) = u_t(x, 0)$ are odd with respect to the reflection through S. If the zero hypersurface S is not a hyperplane, this strong symmetry principle does not hold. Nevertheless, we will show that a weaker symmetry principle holds in the nonlinear case as well. This fact will be a key to the proof that stationary surfaces are ruled if the support of f is convex.

DEFINITION 2.4. Let Π be a hyperplane in $\mathbb{R}^n$ and let x and y be points in $\mathbb{R}^n$, then x and y are $\Pi-$*mirror* if and only if they are mirror images with respect to Π.

If A is a hypersurface in $\mathbb{R}^n$ and $a \in A$ then $T_a(A)$ will denote the hyperplane tangent to A at a.

The following theorem is the key microlocal result.

THEOREM 2.5 (Theorem 3.2 [**5**], Microlocal Symmetry Theorem). *Let A be a real-analytic hypersurface in $\mathbb{R}^n$, and let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution in $\mathbb{R}^n$. Let $a_0 \in A \setminus \operatorname{supp} f$, and let $r_0 > 0$ be chosen so $S(a_0, r_0)$ is the smallest sphere centered at a_0 touching* $\operatorname{supp} f$. *Assume $Rf(a,r) = 0$ for all $a \in A$ and all r in an open neighborhood of r_0. If $x \in \operatorname{supp} f \cap S(a_0, r_0)$, then its $T_{a_0}(A)-$mirror point is also in* $\operatorname{supp} f$. *Therefore, if $S(a_0, r_0)$ meets* $\operatorname{supp} f$ *at only one point, this point must be on $T_{a_0}(A)$.*

This theorem is a generalization of a theorem of Courant and Hilbert ([**10**], p. 699 *ff.*) which is equivalent to the reflection principle for the wave equation and states that if $A = \Pi$ is a hyperplane and $Rf(a,r) = 0$ for all $a \in \Pi$ and $r > 0$, then f is odd about Π and so $\operatorname{supp} f$ is *globally* symmetric about Π. For Theorem 2.5, A can be curved, and in this case, at least some points in $\operatorname{supp} f$ are reflections about tangent planes to A.

PROOF OF THEOREM 2.3. Let a_0 be a regular point of $S[f]$ outside E. Since $S[f]$ is the intersection of zero sets of polynomials, (e.g., [**3**]), and a_0 is a regular point, there is a real-analytic surface $A \subset S[f]$ such that $a_0 \in A$. Since a_0 is outside E, the sphere $S(a_0, r_0)$ of smallest radius centered at a_0 that meets $\operatorname{supp} f$ has positive radius. Since E is convex, this sphere meets E at only one point, x_0. By Microlocal Symmetry Theorem 2.5, $x_0 \in T_{a_0}(A)$. Since this is true for every point on A near a_0, the segment from a_0 to x_0 must be in $S[f]$. Since $S[f]$ is an algebraic surface, the entire line, ℓ_0 from a_0 to x_0 is in $S[f]$.

The line ℓ_0 is orthogonal to E in the sense of Definition 2.2 since E is convex and the sphere $S(a_0, r_0)$ meets E only at the one point x_0 and $x_0 \in \ell_0$.

Now let a_0 be a singular point of S that is outside E. Because S is algebraic, there exists a sequence of regular points in S $\{a_j\}$ that converges to a_0.

Since a_j is a regular point, we now know there is a line ℓ_j in S through a_j and orthogonal to E. Let x_j be the point of intersection in $\ell_j \cap E$ that is closer to a_j. Since E is compact, there exists a subsequence $x_{j_k} \to x_0$ for some $x_0 \in E$. Let ℓ_0 be the line through a_0 and x_0. Because S is closed, $a_{j_k} \to a_0$, and $x_{j_k} \to x_0$, ℓ_0 must be in S. Since the ℓ_j are all orthogonal to E, ℓ_0 must be orthogonal to E in the sense of Definition 2.2. This proves the theorem when a_0 is singular. □

3. Some observations supporting Conjecture 2.1.

The following support theorem was proved in [**3**] for the circle transform, and the proof is valid in $\mathbb{R}^n$ since the microlocal analysis is the same and the analogous geometric arguments are valid.

We call two points u and v in a smooth hypersurface $S \subset \mathbb{R}^n$ *opposite* if the segment $[u, v]$ is perpendicular to the both tangent planes $T_u(S)$ and $T_v(S)$.

THEOREM 3.1 (Support Theorem 4.1 [**3**]). *If $f \in C_c(\mathbb{R}^n)$ and the stationary set $S[f]$ contains at least two (regular) opposite points then $f = 0$.*

The proof follows by applying Microlocal Symmetry Theorem 2.5 to the tangent planes $T_a(S)$ and $T_b(S)$ to erase the support of f.

Now assume that supp f is bounded by a smooth closed convex surface E. The stationary surface can be represented as $S[f] = V \cup S$, where V is an algebraic variety of codimension greater than 1 and S is an algebraic hypersurface [**3**]. Theorem 2.3 says that in our case S is ruled, that is S is a union of straight lines, orthogonal to the surface E at both intersection points. To verify Conjecture 2.1 we would need to prove that S is a cone, that is to prove that all ruling lines intersect at one point, and we are not able to do this at the moment.

Nevertheless, the above theorem about opposite points allow us to prove some observations that support the conjecture. We consider a smooth line bundle $\pi : X \mapsto B$. By a smooth line bundle we understand the triple (X, B, π) where X and B are smooth manifolds in $\mathbb{R}^n$ (X is the space of the bundle and B is its base), π is a smooth map, and the preimages $\pi^{-1}(b), b \in B$ are lines in $\mathbb{R}^n$. If $b(t)$ is a local parametrization for the base B and λ is the natural parameter on the line $\pi^{-1}(b(t))$, then (t, λ) are local coordinates on the manifold X. We further assume this imbedding is globally regular, so (t, λ) give coordinates on X for $\lambda \in \mathbb{R}$.

THEOREM 3.2. *Let $n > 2$ and let X be a smooth hypersurface which is the space of a line bundle (X, B, π) in $\mathbb{R}^n$ with the base B a smooth closed compact manifold. Assume that any two lines in X that are maximally far apart are not parallel. Then X is contained in no stationary set $S[f]$ for the wave equation with nonzero compactly supported initial data f.*

We exclude $n = 2$ in the statement of the theorem since, in this case, B is a discrete collection of points.

PROOF. Since B is compact and the line fibration of X is continuous, there are two lines $L_0 = \pi^{-1}(b_0),\ L_1 = \pi^{-1}(b_1)$ of maximal (positive) distance between them. Let $u_0 \in L_0,\ u_1 \in L_1$ be the closest points in these lines. Then the segment $[u_0, u_1]$ is orthogonal to both lines L_0 and L_1.

Moreover, since the distance $dist(L, M)$ between any two line fibers attains its maximum at $L = L_0, M = L_1$ and the fibration is smooth, we will prove that the segment $[u_0, u_1]$ is orthogonal to the tangent planes $T_{u_0}(X)$ and $T_{u_1}(X)$. To prove this, we now specify coordinates on B and X near b_0 and b_1.

The dimension of the base B equals $n - 2$ as $\dim X = n - 1$ and the fibers are one-dimensional. We can find disjoint open sets in $\mathbb{R}^{n-2}$, U_0 and U_1 and a map $b : U_0 \cup U_1 \to B$ that give coordinates for B near b_0 and b_1 so that if $s \in U_0$, then $b(s)$ gives coordinates near b_0 and for $t \in U_1$, then $b(t)$ gives coordinates for B near b_1 We let $b(s_0) = b_0$ and $b(t_0) = b_1$.

This gives coordinates on the set of lines in X near L_0 and near L_1:

$$\text{for } s \in U_0, L(s) = \pi^{-1}(b(s)),\ \text{for } t \in U_1, L(t) = \pi^{-1}(b(t))\,.$$

We define $\nu : U_0 \cup U_1 \to S^{n-1}$ to be a unit direction vector for the line $L(s)$ for $s \in U_0$ and a unit direction vector for $L(t)$ for $t \in U_1$. If the coordinate patches on

B are sufficiently small, ν can be assumed to be well defined and smooth. Perhaps by shrinking U_1 and U_2 further, we can assume that no lines $L(s)$ and $L(t)$ for $s \in U_1$ and $t \in U_2$ are parallel (since L_1 and L_2 aren't parallel by assumption).

Since the lines $L(s)$ and $L(t)$ are not parallel, for $(s,t) \in U_0 \times U_1$, there are uniquely determined points $u_0(s,t) \in L(s)$ and $u_1(s,t) \in L(t)$ such that

$$\operatorname{dist}(u_0(s,t), u_1(s,t)) = \operatorname{dist}(L(s), L(t)),$$

and the dependence on (s,t) is smooth. Note that $u_0(s_0,t_0) = u_0,\ u_1(s_0,t_0) = u_1$. There are smooth functions λ_0 and λ_1 on $U_0 \times U_1$ such that

$$u_0(s,t) = b(s) + \lambda_0(s,t)\nu(s), \quad u_1(s,t) = b(t) + \lambda_1(s,t)\nu(t)\,. \tag{3.1}$$

The function

$$\rho(s,t) = \|u_0(s,t) - u_1(s,t)\|^2$$

attains its maximum at (s_0,t_0), so $\partial_{s_k}\rho(s_0,t_0) = 0$, where ∂_{s_k} is the partial differentiation in the k^{th} coordinate of s, $k \in \{1,\dots,n-2\}$. If we take this partial derivative at (s,t) and use (3.1), we get

$$\langle u_0 - u_1, (\partial_{s_k} b)(s) + (\partial_{s_k}\lambda_0)(s,t)\nu(s) + \lambda_0(s,t)(\partial_{s_k}\nu)(s) - (\partial_{s_k}\lambda_1(s,t))\nu(t)\rangle\,. \tag{3.2}$$

If we evaluate (3.2) at (s_0,t_0) and use the fact that $\nu(s_0)$ and $\nu(t_0)$ are both orthogonal to the segment $[u_0(s_0,t_0), u_1(s_0,t_0)]$, we get

$$\langle u_0(s_0,t_0) - u_1(s_0,t_0), (\partial_{s_k} b)(s_0) + \lambda_0(s_0,t_0)(\partial_{s_k}\nu)(s_0)\rangle = 0\,.$$

Note that the map $U_0 \times \mathbb{R} \ni (s,\lambda) \to b(s) + (\lambda + \lambda_0(s_0,t_0))\nu(s)$ gives local coordinates on X near u_0. (Here we use the assumption that the fibration is globally regular.) This implies

$$\begin{aligned} T_{u_0}X = \operatorname{span}\{&\nu(s_0), (\partial_{s_1} b)(s_0) + \lambda_0(s_0,t_0)\partial_{s_1}\nu(s_0), \dots, \\ &(\partial_{s_{n-2}} b)(s_0) + \lambda_0(s_0,t_0)\partial_{s_{n-2}}\nu(s_0)\}, \end{aligned}$$

Therefore, we obtain that the segment $[u_0,u_1]$ is perpendicular to $T_{u_0}X$. Orthogonality to the second tangent plane T_{u_1} is obtained by differentiation in the parameter t. This shows that u_1 and u_2 are opposite points, and we can use the support Theorem 3.1 to conclude X can not be the stationary set of any nonzero $f \in C_c(\mathbb{R}^n)$. □

We believe that the requirement that X contains no parallel lines is technical. If the lines of maximal distance are parallel, then we need to know that there are smooth maps $u_0(s,t)$ and $u_1(s,t)$ near (s_0,t_0).

As was stated in Theorem 2.3, when the outer boundary of $\operatorname{supp} f$ is convex, the stationary set contains a surface built up of lines. The set of these lines is compact because all of them meet the outer boundary E of the support. Theorem 3.2 says that the union of the lines cannot constitute a smooth fibration. So, the fibration must have singular points, as is true for cones. Therefore, this assertion is in the direction of the expected fact that $S[f]$ is a cone but, of course, is pretty far from this conclusion. Let us investigate the case of convex support in more detail.

Theorem 3.3. *Let $n > 2$. Assume the outer boundary, E, of $\operatorname{supp} f$ is a C^2 convex surface. Let S' be the set of lines in $S[f]$ that contain regular points. Then, the lines in each connected component of S' all meet E in segments of the same length.*

For $n = 2$, $S[f]$ consists of, at most, a finite number of lines, and so in Theorem 3.3 there is only one segment in $S[f]$ in each connected component of the set of lines in $S[f]$, hence we exclude this case.

PROOF. Let $L \in S'$ and let $x \in L$ be a regular point. Because $S[f]$ is regular near x, there is a C^2 hypersurface $A \subset S[f]$ with $x \in A$. Since E is C^2 and A and E intersect at right angles, the $n-2$ surface $F = E \cap A$ is regular. Let $a < 0 < b$ and let $\gamma : (a, b) \to F$ be a differentiable curve through $x = \gamma(0)$ and for $t \in (a, b)$ let $\nu(t)$ be the outer unit normal to E at $\gamma(t)$. Then,

$$\gamma'(t) \cdot \nu(t) = 0, \qquad \nu'(t) \cdot \nu(t) = 0\,. \tag{3.3}$$

For $t \in (a, b)$ let $\ell(t)$ be the line through $\gamma(t)$ and parallel to $\nu(t)$. Then, by Theorem 2.3, $\ell(t) \subset S[f]$. Let $r(t)$ be the length of the segment of $\ell(t)$ inside E. Then, the points of intersection of $\ell(t)$ with E are $\gamma(t)$ and

$$\widetilde{\gamma}(t) = \gamma(t) - r(t)\nu(t).$$

We show this distance $r(t)$ is constant near $t = 0$, and that will prove the theorem.

Since $\widetilde{\gamma}(t)$ parameterizes a curve on the other side of E and the line $\ell(t)$ is perpendicular to E at both points of intersection, $\gamma(t)$ and $\widetilde{\gamma}(t)$, $\nu(t)$ is perpendicular to both curves γ and $\widetilde{\gamma}$ at t. Therefore,

$$\nu(t) \cdot \widetilde{\gamma}'(t) = \nu(t) \cdot (\gamma'(t) - r(t)\nu'(t) - r'(t)\nu(t)) = 0\,,$$

and using (3.3), we get

$$-r'(t) = 0\,.$$

This shows the length, $r(t)$, of the segment of $\ell(t)$ inside E is constant.

If there is another point on L that is on another regular surface A' in $S[f]$, then the same proof shows that lines in S' near L and defined by this surface also meet E in segments of the same length, the length of the segment of L inside E. So, all lines in S' sufficiently near L have this property. Therefore, all lines in the same component of S' containing L have this property. □

Conjecture 2.1 is easy to check for some particular cases when the outer boundary of $\operatorname{supp} f$, E, is quadratic, that is E is an ellipsoid.

THEOREM 3.4. *Let E be the ellipsoid in $\mathbb{R}^n$:*

$$E = \{\sum_{j=1}^{n} (x^2/\alpha_j) = 1\},\ 0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n.$$

Let k be the number of equal coefficients α_j. Then the hypersurface part $S \subset S[f]$ is empty if $k \le n-2$, and is a cone if $k = n-1$ or $k = n$, and moreover S is a coordinate hyperplane when $k = n-1$.

PROOF. We know that S is a ruled surface built from lines intersecting E orthogonally at both intersection points in $S \cap E$. Take a ruling line L in S, $L = \{x = u + \lambda\nu \,\big|\, \lambda \in \mathbb{R}\}$, where $u \in E \cap S$. Let $v = u + \lambda_0\nu$ be the second intersection point of L with the ellipsoid E. We know that L is orthogonal to E at both points u and v, hence ν is proportional to the normal vectors

$$n_u = (u_1/\alpha_1, \cdots u_n/\alpha_n)$$

and

$$n_v = ((u_1 + \lambda_0\nu_1)/\alpha_1, \cdots, (u_n + \lambda_0\nu_n)/\alpha_n)$$

to the ellipsoid E at the points u and v respectively. Then the difference of the normal vectors is parallel to ν as well, and

$$(\nu_1, \cdots, \nu_n) = c(\nu_1/\alpha_1, \cdots \nu_n/\alpha_n)$$

with some nonzero constant c. Therefore $\nu_j = 0$ unless $\alpha_j = 1/c$. Thus, if all α_j are different then S consists of a single line which is impossible as S is $(n-1)-$dimensional. If k coefficients α_j coincide then at most k certain coordinates of the vector ν are nonzero and therefore all lines constituting S belong to a $(n-k)-$dimensional coordinate plane. When $k = n-1$ we have that S is a coordinate hyperplane. Finally, $k = n$ means that E is a sphere and since all ruling lines are orthogonal to E we conclude that S is a cone centered at the origin. □

4. Proofs of Theorems about $S_T[f]$

Proof of Theorem 1.1. The key to the proof is the Poisson-Kirchhoff formula (see e.g., [**14**, p. 130, (i)], [**10**, p. 682]) for the solution $u(x,t)$ of the Cauchy problem (1.1):

$$u(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}F(x,t) \tag{4.1}$$

$$\text{where}\quad F(x,t) = \int_0^t (t^2 - r^2)^{(n-3)/2} rRf(x,r)dr\,. \tag{4.2}$$

First we consider the case when n is even. The basic idea of the proof is to use the Poisson-Kirchhoff formula and the compact support of f to show that if $x \in S_T[f]$, $Rf(x,r) = 0$ for all $r > 0$. We show $Rf(x,\cdot)$ is zero by showing most moments in r of $Rf(x,r)$ are zero.

Let $x \in S_T[f]$ so $u(x,t) = 0$ for $t > T = T(x)$, where u is the solution to (1.1) for f. Then for $t > T(x)$, $F(x,t)$ in (4.2) is a polynomial of degree at most $n-3$. We divide the expression for $F(x,t)$ by t^{n-3} and we get that

$$\int_0^t (1 - (r/t)^2)^{(n-3)/2} rRf(x,r)\,dr = \sum_{k=0}^{n-3} c_k t^{-k}\,. \tag{4.3}$$

For some $T' \geq T(x)$ we know that $\operatorname{supp} f$ lies inside the ball centered at x of radius T'. Using this information, we take $t > T'$ and expand $(1-(r/t)^2)^{(n-3)/2}$ in powers of $(r/t)^2$ and equate to the finite sum on the right of (4.3),

$$\sum_{j=0}^{\infty} t^{-2j} \int_0^{T'} d_j r^{2j} rRf(x,r)\,dr = \sum_{k=0}^{n-3} c_k t^{-k}\,, \tag{4.4}$$

where the d_j are all nonzero by the Binomial Theorem since $(n-3)/2 \notin \mathbb{N}$. This shows that all even moments of $rRf(x,r)$ of order greater than $n-3$ are zero and this shows $Rf(x,r) = 0$ for all $r > 0$. So, $S_T[f] \subset N[f]$.

If $x \in N[f]$ then $x \in S[f]$ by just plugging $Rf(x,r) = 0$ into the Poisson Kirchhoff Formula (4.1). This shows $S_T[f] \subset N[f] = S[f] \subset S_T[f]$ so $S_T[f] = N[f] = S[f]$.

To see $S_T[f]$ can be a proper subset of $N_T[f]$ for n even, let f be a non-zero, non-negative, smooth radial function supported in the unit ball. Let $T = 1$. Then $0 \in N_1[f]$. However, $0 \notin N[f]$ since $Rf(0,t)$ is nonzero for some $t \in (0,1)$. However, we just showed $N[f] = S_1[f]$, so this shows $S_1[f]$ is a proper subset of $N_1[f]$ in this case.

Now, let's consider n odd. In this case, the Poisson-Kirchhoff formula can be rewritten [**10**, p. 682] as

$$u(x,t) = \frac{\sqrt{\pi}}{2\Gamma(n/2)} \left(\frac{\partial}{2t\partial t} \right)^{(n-3)/2} \left(t^{n-2} Rf(x,t) \right) \tag{4.5}$$

Since $x \in S_T[f]$, $\left(\frac{\partial}{2t\partial t}\right)^{(n-3)/2} \left(t^{n-2}Rf(x,t)\right) = 0$ for $t > T(x)$. It is a straightforward exercise to show this implies $Rf(x,t)$ is a polynomial in t^2 of degree less than $(n-3)/2$ for $t > T(x)$. However, since f has compact support, $Rf(x,t) = 0$ for large t therefore this polynomial is zero for $t > T(x)$ and $S_T[f] \subset N_T[f]$. The reverse containment is proven by plugging $Rf(x,t) = 0$ into (4.5). □

For n even, we just proved $S_T[f] = N[f] = S[f]$, but for n odd, $S_T[f] = N_T[f]$, that is $x \in S_T[f]$ if and only if $Rf(x,r) = 0$ for all $r > T(x)$. Even in this case, we can say something about the support of f.

Let S be a smooth hypersurface (not necessarily connected) in $\mathbb{R}^n$. Recall that two distinct points $a, b \in S$ *opposite* if the segment $[a,b]$ is orthogonal to both affine tangent planes $T_a(S)$ and $T_b(S)$ to S at the points a and b respectively.

THEOREM 4.1. *Let n be odd and let $T_0 > 0$ be constant. Assume $S_{T_0}[f]$ contains two regular opposite points a and b and a manifold $A \subset S_{T_0}[f]$ that makes them regular is real-analytic. Then,* $\operatorname{supp} f$ *is enclosed in both spheres $S(a,T_0)$ and $S(b,T_0)$.*

Note that this opposite point condition trivially applies if $S[f]$ contains an open set, U, because then A can be taken as two parallel hyperplanes sections of U. This observation justifies Theorem 1.3.

PROOF. We prove this by contradiction. Assume $\operatorname{supp} f$ is not zero outside both spheres $S(a,T_0)$ and $S(b,T_0)$ and let $r_a \geq T_0$ and $r_b \geq T_0$ such that the sphere $S(a,r_a)$ is the smallest sphere centered at a enclosing $\operatorname{supp} f$ and $S(b,r_b)$ is the smallest sphere centered at b enclosing $\operatorname{supp} f$. We can assume $r_a \geq r_b$, and by assumption, $r_a > T_0$. Let A be a real-analytic hypersurface in $S[f]$ (not necessarily connected) containing a and b. By Theorem 1.1, we have that $Rf(x,r)$ is zero for $r > T_0$ and $x \in S_{T_0}[f]$, and we know that a and b are opposite points.

By the Microlocal Symmetry Theorem 2.5 all points on $S(b,r_b) \cap \operatorname{supp} f$ are $T_b(A)$ symmetric (unless $r_b = T_0$), and all points on $S(a,r_a) \cap \operatorname{supp} f$ are $T_a(A)$ symmetric. Because a and b are opposite and $r_a \geq r_b$, some of these points on $S(a,r_a)$ must be outside $S(b,r_b)$ and therefore not in $\operatorname{supp} f$. This contradiction proves the theorem. □

It should be pointed out that this argument is analogous to the one that we used to prove the support theorem, Theorem 4.1 in [**3**].

Acknowledgments. The authors thank Henry Warchall for asking the intriguing question what we could say about stationary sets for time $t > T_0$. The authors thank Peter Kuchment and Jan Boman for stimulating conversations about this research over the years.

References

1. Gaik Ambartsoumian and Peter Kuchment, On the injectivity of the circular Radon transform arising in thermoacoustic tomography, *Inverse Problems* **21**, 473-485.

2. M. Agranovsky, C. Berenstein, and P. Kuchment *Approximation by spherical waves in L^p spaces*, J. Geom. Analysis **6** (1996), 365-383.
3. M. Agranovsky and ETC. Quinto, *Injectivity sets for the Radon transform over circles and complete systems of radial functions*, J. Funct. Anal. **139** (2), (1996), 383-414.
4. M. Agranovsky and E.T. Quinto, *Injectivity of the spherical mean operator and related problems*, in: Complex Analysis, Harmonic Analysis and Applications (R. Deville et al, eds.), Addison Wesley, London, 1996.
5. M. Agranovsky and E.T. Quinto, *Geometry of stationary sets for the wave equation in $\mathbb{R}^n$. The case of finitely supported initial data*, Duke Math. J. **107** (2000), 57-84.
6. M. Agranovsky and E.T. Quinto, *Stationary Sets for the wave equation in crystallographic domains*, Trans. Amer. Math. Soc., **355** (2003), 2439-2451.
7. M. Agranovsky, V.V. Volchkov and L. Zalcman, *Conical Uniqueness Sets for the Spherical Radon Transform*, Bull. London Math. Soc., **31**(1999), 231-236.
8. C. Bar, *On nodal sets for Dirac and Laplace operators*, Comm. Math. Phys., **188** (1997), 709-721.
9. S. Y. Cheng, *Eigenfunctions and nodal sets*, Comm. Math. Helv., **57** (1976), 43-55.
10. Courant R. and Hilbert D., *Methods of Mathematical Physics, vol. 2*, Interscience, New York, 1953.
11. H. Donelly and C. Fefferman, *Nodal sets of eigenfunction on Riemannian manifolds*, Invent. Math. **93** (1988), 161-183.
12. ———, *Nodal sets of eigenfunction on surfaces*, J. Amer. Math. Soc., **3** (1990), 333-353.
13. David Finch, Sarah Patch, and Rakesh, *Determining a function from its mean values over a family of spheres* SIAM J. Math. Anal., **35**(2004), 1213-1240.
14. S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, 1984.
15. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer Verlag, New York, 1983.
16. V. Ya. Lin and A. Pinkus, *Approximation of multivariate functions*, in Advances in Computational Mathematics, editors H.P. Dikshit and C.A. Micchelli, World Scientific Publishers, 1994, pp. 1–9.
17. E.T. Quinto, *Support Theorems for the Spherical Radon Transform on Manifolds*, preprint, 2005.

Bar Ilan University, Ramat Gan, Israel
E-mail address: agranovs@macs.biu.ac.il

Tufts University, Medford, MA, USA
E-mail address: todd.quinto@tufts.edu