INJECTIVITY OF THE SPHERICAL MEAN OPERATOR AND RELATED PROBLEMS

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ABSTRACT. The problem of injectivity for a Radon transform over level sets of polynomials in $\mathbb{R}^n$ is studied. The main results concern the spherical mean operator defined on compactly supported continuous functions. Related problems and more general transforms are discussed.

0. INTRODUCTION

Integral geometry deals with the reconstruction of functions, measures, or distributions from their integrals over certain geometric objects. Various problems in analysis lead to integration over level sets of polynomials, *i.e.* algebraic curves or surfaces, and therefore, to the corresponding Radon transform. The classical example is the usual Radon transform over lines or planes and this corresponds to (rotations and translations of) polynomials degree 1. We will first give the general set up, then carefully look at a specific example, the spherical mean transform and discuss applications. Then, we will outline the proofs of our results. Finally, we give results and conjectures for higher dimensional spaces.

Let P be a given polynomial in $\mathbb{R}^n$ with real coefficients. We will associate a Radon transform, R_P, on functions f belonging to some linear subspace F of the space $C(\mathbb{R}^n)$ of all continuous functions in $\mathbb{R}^n$ (or of the dual space $(C(\mathbb{R}^n))'$ in (1e)). The main case will be $F = C_c(\mathbb{R}^n)$, the subspace of $C(\mathbb{R}^n)$ consisting of functions with compact support.

To define this transform we need sets to integrate over and measures of integration. The sets will be the algebraic varieties:

(1a) $$M_{x,\alpha} = \{\xi \in \mathbb{R}^n \mid P(\xi - x) = \alpha\},\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}.$$

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A natural choice of measure on $M_{x,\alpha}$ is the measure $\mu_{x,\alpha}$ defined as follows. For each $x \in \mathbb{R}^n$ we represent $\mathbb{R}^n$ as the union of sheets of the spread

$$\Xi_x = \{M_{x,\alpha}\}_{\alpha\in\mathbb{R}},\ \mathbb{R}^n = \bigcup_{\alpha\in\mathbb{R}} M_{x,\alpha}. \tag{1b}$$

We choose measures $\mu_{x,\alpha}$ so that for any $x \in \mathbb{R}^n$ the decomposition

(1c) $\qquad d\xi = d\mu_{x,\alpha}(\xi) \times dv(\alpha)$ holds for an appropriate measure $dv(\alpha)$.

Now, we define the Radon transform, R_P for $f \in F$ by

$$R_P f(x,\alpha) = \int_{P(\xi-x)=\alpha} f(\xi) d\mu_{x,\alpha}(\xi) \tag{1d}$$

The integral transform (1d) with the measure from (1c) is considered in the book [E] of Ehrenpreis in connection with the nonlinear Fourier transform.

This transform can easily be defined as a transform from $C'(\mathbb{R}^n)$ to the set of functions from $\mathbb{R}^n$ to measures on $\mathbb{R}$. For each $x \in \mathbb{R}^n$, $R_P(x,\cdot)$ is the measure on $\mathbb{R}$ defined for $g \in C(\mathbb{R})$ by

$$\int_{\alpha\in\mathbb{R}} g(\alpha) dR_P\mu(x,\alpha) :\ = \int_{\mathbb{R}^n} g(P(\xi - x)) d\mu(\xi). \tag{1e}$$

If $\mu = f(\xi)d\xi$ with $f \in C_c(\mathbb{R}^n)$, this definition is consistent with (1a-d); the measure corresponding to $R_P f$ is $(R_P f(x,\alpha))d\nu(\alpha)$ where ν is the measure in (1c). If Γ is a manifold and P is sufficiently regular so that R_P can be defined using the double fibration [He1, GS] with consistent smooth manifolds $M_{x,\alpha}$ and consistent smooth measures, R_P can be defined on distributions in a consistent way with (1a-e).

Now we are going to formulate the main problem. Let $\Gamma \subset \mathbb{R}^n$. Formula (1) defines the linear operator

$$R_P : F \to C(\Gamma \times \mathbb{R}) \tag{2}$$

Definition. A set Γ is said to be a *set of injectivity* (for the transform R_P on the space F), or, transform R_p is *injective on* Γ, if and only if the operator (2) is injective.

In other words R_P is injective on Γ (on domain F) if, whenever $R_P f(x,\alpha) = 0$ for all $f \in F$ and all $(x,\alpha) \in \Gamma \times \mathbb{R}$, then $f = 0$. The main subject of this paper is the following.

Problem 1. *Given a polynomial P and a linear space $F \subset C(\mathbb{R}^n)$, describe sets of injectivity of the transform R_P on F.*

This problem is also mentioned in the Ehrenpreis book [E]. The kernel of the transform R_P on a set Γ is defined by:

$$\ker_\Gamma R_P = \{f \in F \mid R_P f(x,\alpha) = 0 \text{ for all } (x,\alpha) \in \Gamma \times \mathbb{R}\} \tag{3}$$

Problem 1 is contained in the more general problem:

Problem 2. *Given a set* $\Gamma \subset \mathbb{R}^n$, *describe* $\ker_\Gamma R_P$.

For any $f \in F$ define the set

$$S[f] = \{x \in \mathbb{R}^n \mid R_P f(x, \alpha) = 0 \text{ for all } \alpha \in \mathbb{R}\}. \tag{4}$$

Clearly, sets of injectivity, Γ, are characterized by the property: $\Gamma \subset S[f]$ implies $f \equiv 0$. Thus the problem can be formulated also as follows:

Problem 3. *Given* $f \in F$, *describe the set* $S[f]$.

1. Equivalent Problems

1.1. Approximation of functions of several variables.

The subject of this section is strongly related to papers of V. Lin and A. Pinkus [LP1, LP2]. Having been motivated by Hilbert's 13th problem on superpositions, they considered the following interesting problem.

Fix a set $\Phi \subset C(\mathbb{R}^n)$ and consider the linear space of finite linear combinations

$$\sum \alpha_i g_i(\varphi_i|x|). \tag{5}$$

where $\alpha_i \in \mathbb{R}$, $g_i \in C(\mathbb{R})$, and $\varphi_i \in \Phi$.

Question. *(Lin-Pinkus* [LP2]*). For which sets* Φ *are finite linear combinations* (5) *dense in the space* $C(\mathbb{R}^n)$ *equipped with the topology of uniform convergence on compact sets?*

As suggested in [LP2], it is natural to choose a fixed function, φ and define Φ to be some set of translations of φ. In turn, it is natural to choose φ to be a polynomial.

Thus, given a polynomial P in $\mathbb{R}^n$ and a set $\Gamma \subset \mathbb{R}^n$, define the linear space $\mathcal{L}(P, \Gamma) = \operatorname{span}\{g(P(x - x_0)) \mid g \in C(\mathbb{R}), x_0 \in \Gamma\}$.

Problem 4. [LP2]. *Given a polynomial* P, *describe all sets* $\Gamma \subset \mathbb{R}^n$ *such that the space* $\mathcal{L}(P, \Gamma)$ *is dense in* $C(\mathbb{R}^n)$.

Obviously, for linear P the space $\mathcal{L}(P, \Gamma)$ is never dense, hence the first interesting case is $\deg P = 2$.

By the Hahn-Banach theorem, Problem 4 is equivalent to finding all sets $\Gamma \subset \mathbb{R}^n$ such that the only compactly supported measure $\mu \in C(\mathbb{R}^n)'$ which annihilates $\mathcal{L}(P, \Gamma)$ is zero ($\mathcal{L}(P, \Gamma)^\perp = \{0\}$).

In general, the condition $\mu \in \mathcal{L}(P, \Gamma)^\perp$ is equivalent to $R_P \mu = 0$ where R_P is defined on measures by (1e). Thus, we arrive at the injectivity problem for the transform R_P defined by (1e). Of course, if $\mu = f(\xi) d\xi$, $f \in C_c(\mathbb{R}^n)$, then $\mu \in \mathcal{L}(P, \Gamma)^\perp$ is equivalent to $R_P f(x, \alpha) = 0$ for all $(x, \alpha) \in \Gamma \times \mathbb{R}$. Moreover, for certain polynomials P, denseness of $\mathcal{L}(P, \Gamma)$ and injectivity *on functions* of the operator R_P and Γ are equivalent. See Proposition 1.1 below.

1.2 Decompositions into Spherical Waves.

Let us consider the simplest and most interesting case (*cf.* [LP2]):

$$P(x) = |x|^2 = x_1^2 + \ldots + x_n^2, x = (x_1, \ldots, x_n).$$

Then, the *Spherical Mean Operator (s.m.o.)* is defined for $f \in C_c(\mathbb{R}^n)$ as in (1) by

$$Rf(x,r) = \int_{S(x,r)} f(\xi) dA(\xi). \tag{6}$$

Here dA is the normalized area measure on the space $S(x,r) = \{\xi \in \mathbb{R}^n : |\xi - x| = r\}$. Note that the s.m.o. satisfies the definition (1) (with $R_P f(x,\alpha) = Rf(x, \sqrt{\alpha})$). For the s.m.o., our next proposition shows that the problems in §1.1 for measures and functions are equivalent.

Proposition 1.1. *Let $\Gamma \subset \mathbb{R}^n$, then the functions, $f \in C_c(\mathbb{R}^n)$ that are in* $\ker_\Gamma R$ *are dense in* $\ker_\Gamma R$ *(where $F = C'(\mathbb{R}^n)$). So, $\Gamma \subset \mathbb{R}^n$ is a set of injectivity for the spherical mean operator R on $C_c(\mathbb{R}^n)$ if and only if Γ is a set of injectivity for R on compactly supported measures.*

The proof is a simple convolution argument: if μ is a measure in $\ker_\Gamma R$, then the function $\mu * \varphi$ is in $\ker_\Gamma R$ for any radial φ. Letting φ be an approximate identity finishes the proof.

By analogy with planar waves, we will call any continuous function f of the form $f(x) = g(|x-a|^2)$, $g \in C(\mathbb{R})$ a *spherical wave* centered at the point $a \in \mathbb{R}^n$. Denote L_a the space of all spherical waves at the point a. Note that $L_a \cap L_b = \{\text{constants}\}$ for $a \neq b$.

The question in [LP2] can be formulated as follows: describe all sets $\Gamma \subset \mathbb{R}^n$ for which

$$C(\mathbb{R}^n) = \mathrm{cl}\left(\bigoplus_{a\in\Gamma} L_a\right). \tag{7a}$$

According to Proposition 1.1, formula (7a) is true if and only if Γ is a set of injectivity for the s.m.o. on functions ($F = C_c(\mathbb{R}^n)$).

Let $\mathcal{S} \subset \mathbb{R}$ be a set with an accumulation point in $\mathbb{R} \cup \{\infty\}$. Since the sets $B = \{|\alpha|^{2k} \mid k = 0, 1\ldots\}$ and $C = \{\exp(\sigma|\alpha|) \mid \sigma \in \mathcal{S}\}$ are each dense in $C(\mathbb{R})$ in the topology of uniform convergence on compact sets, one can rephrase (7a): describe all sets $\Gamma \subset \mathbb{R}^n$ for which

$$C(\mathbb{R}^n) = \mathrm{cl}\left(\bigoplus_{a\in\Gamma} \mathrm{span}\left\{|x-a|^{2k} \mid k = 0, 1, \ldots\right\}\right) \tag{7b}$$

or equivalently describe all sets $\Gamma \subset \mathbb{R}^n$ for which

$$C(\mathbb{R}^n) = \mathrm{cl}\left(\bigoplus_{a\in\Gamma} \mathrm{span}\left\{\exp(\sigma|x-a|) \mid \sigma \in \mathcal{S}\right\}\right) \tag{7c}$$

According to Proposition 1.1, formula (7a) is true if and only if Γ is a set of injectivity for the *Spherical Mean Operator (s.m.o.)* defined on the space $C_c(\mathbb{R}^n)$. In [LP2], some examples are given when the decomposition (7a) holds ($n = 2,\ \Gamma$ is an ellipse or parabola) and when it does not (Γ is a straight line).

The following example of a set Γ for which the decomposition (7a) fails is mentioned in [LP2, p.5]: Γ is a set of lines with a common intersection point and such that the angles between each of the lines are rational multiples of π. Lin and Pinkus independently conjectured [L] that these sets essentially are the only sets Γ for those (7a) does not take place. Before having learned of this conjecture and the example in [LP2], the first author conjectured this, and both authors proved it in [AQ1], [AQ2] (see §4).

1.3. Uniqueness Theorems for PDE.

Darboux equation. Due to a theorem of Asgiersson, the spherical means

$$u(x,r) = Rf(x,r) = \int_{|\xi - x| = r} f(\xi)\,dA(\xi)$$

satisfy the Darboux equation for $u = u(x,r),\ (x,r) \in \mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{aligned} u_{rr} + \left(\frac{n-1}{r}\right)u_r - \Delta_x u = 0,\\ u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R}^n. \end{aligned} \tag{8}$$

Therefore, the problem of injectivity is equivalent to uniqueness of solutions of (8). Namely, we are looking for such sets $\Gamma \subset \mathbb{R}^n$ such that if u is a solution of (8) with Cauchy data $f \in C_c(\mathbb{R}^n)$ and $u|_{\Gamma\times\mathbb{R}} = 0$ then $u \equiv 0$.

Heat equation. Consider the Cauchy problem for the heat equation in $\mathbb{R}^n$ where $u = u(x,t),\ (x,t) \in \mathbb{R}^n \times [0,T],\ T > 0$:

$$\begin{aligned} u_t = c^2 \Delta u,\\ u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R}^n. \end{aligned}$$

Let $N[f]$ be the nodal set, the set of points where the temperature is zero all the time:

$$N[f] = \{x \in \mathbb{R}^n \bigm| u(x,t) = 0\ \forall t \in [0,T]\}. \tag{9}$$

The Poisson formula

$$u(x,t) = \frac{1}{t^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{-|x-\xi|^2}{c^2 t}\right) f(\xi)\,d\xi$$

yields that

$$x \in N[f] \quad \text{is equivalent to } Rf(x,\eta) = 0 \quad \text{for all } \eta > 0.$$

Therefore $N[f] = S[f]$, where the set $S[f]$ is defined by formula (4).

Wave equation. P. Kuchment [Ku] observed relation between of the injectivity problem for s.m.o. and wave equation. Let us consider the Cauchy problem for the wave equation for $u = u(x,t),\ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{aligned} u_{tt} &= c^2 \Delta u \\ u(x,0) &= 0, \\ u_t(x,0) &= f(x) \ \text{ for } x \in \mathbb{R}^n. \end{aligned} \tag{10}$$

The Poisson-Kirchoff formula for the solution u implies, as in the case of the heat equation, the identity $N[f] = S[f]$, where $N[f]$ is again the nodal set $N[f] = \{x \in \mathbb{R}^n \bigm| u(x,t) = 0\ \forall t \in \mathbb{R}_+\}$

In the case $n = 2$ the Cauchy problem (10) describes the oscillation of the infinite membrane when it is flat at the initial moment $t = 0$. The nodal set $N[f]$ is the set of stationary points (which do not oscillate). Thus, the injectivity problem for the s.m.o. , R, is equivalent to describing sets of stationary points of the oscillating membrane.

The problems above are discussed to show that a number of interesting questions are closely related to injectivity of the s.m.o. Other results will be mentioned below.

2. Algebraicity of the set $S[f]$. Necessary conditions of injectivity of the transform R_P for $\deg P = 2$

2.1. One formulation of the problem, given in the Introduction, concerns the set $S[f]$, defined by formula (4). Let P be a polynomial. The following simple but very useful observation, due to Lin and Pinkus, shows that if $f \in C_c(\mathbb{R}^n), f \not\equiv 0$ then the set $S[f]$ is algebraic. Indeed, for any function $g \in C(\mathbb{R})$ and all $x \in S[f]$ we have, according to Proposition 1.1:

$$\int_{\mathbb{R}^n} g(P(\xi - x))f(\xi)d\xi = 0.$$

Since f has compact support, the Stone-Weierstrass Theorem implies that it suffices to takes $g(t) = t^k, k = 0, 1, \dots$, and therefore

$$S[f] = \bigcap_{k=0}^{\infty} \left(Q_k[f]\right)^{-1}(0), \tag{11}$$

where $Q_k[f] = P^k * f$ is a polynomial of degree $\deg Q_k[f] \le k \deg P$.

It is clear also, that not all of the polynomials $Q_k[f]$ are identically zero (otherwise $f \equiv 0$).

2.2. Now consider the case deg. $p = 2$. We assume that P is homogeneous,

$$P(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

and $\det(a_{ij}) \neq 0$.

Associate with P another polynomial

$$P^*(x) = \sum_{i,v=1}^{n} a^*_{ij} x_i x_j,$$

where $(a^*_{ij})^n_{i,j=1}$ is the inverse matrix $(a^*_{ij}) = (a_{ij})^{-1}$.

The following identity can be checked by a straight forward computation:

$$P^*(D)P^k = c_k P^{k-1}, c_k = k(k+n-1), \quad \text{where} \quad D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}).$$

This implies

$$P^*(D)Q_k[f] = c_k Q_{k-1}[f]$$

and since not all $Q_k[f]$ are zero, then the nontrivial polynomial $h = Q_{k_{\min}}$ of minimal degree is annihilated by the differential operator $P^*(D)$, namely, $P^*(D)h = cQ_{(k_{\min}-1)}[f] \equiv 0$. Observe that these arguments are true if f is a rapidly decreasing function or even if f is a compactly supported measure. Thus we have arrived at the following necessary condition for injectivity:

Proposition 2.1. *Let Γ be a set of uniqueness for polynomial solutions of the differential equation*

$$P^*(D)h = 0.$$

Then Γ is a set of injectivity for R_P, defined on $C_c(\mathbb{R}^n)$ or, more generally, on the space of rapidly decreasing functions in $\mathbb{R}^n$.

Corollary 2.2. *The s.m.o. is injective on any uniqueness set for harmonic polynomials.*

This fact was first observed by V. Lin and N. Zobin [LZ]. For instance, any closed surface in $\mathbb{R}^n$ or, more generally, any compact $K \subset \mathbb{R}^n$ such that the complement to K has nonempty bounded connected component, is a set of injectivity for the s.m.o., R, on rapidly decreasing functions.

We are able to prove that the last fact is true for the operator R on a large domain, namely, on the space $L^q(\mathbb{R}^n)$ when q is not too large. Correspondingly, the decomposition (7a) is true for $L^p(\mathbb{R}^n)$ if Γ is a closed surface and p is large enough. We are going to return to this subject elsewhere.

3. The injectivity problem for the spherical mean operator

Now we are going to concentrate on the spherical mean operator (s.m.o.):

$$R = R_P, \ P(x) = x_1^2 + \ldots + x_n^2.$$

The s.m.o. plays an important role in analysis and is investigated from various points of view. It is enough to mention the mean value property for harmonic functions and its generalization, the two radii Delsarte theorem [DL, BG, Z3]. In [Z1] an analogous two-radii criteria for analytic functions is obtained. Further generalizations concern tests of harmonicity or analyticity on the hyperbolic disk [BZ, A1, BBHW], the Pompeiu property on symmetric spaces and the Heisenberg group [ABCP, BZ1, BZ3, Th], and the description of $CR-$functions on the Heisenberg group [ABC, Th] (see also the book [A2] and the bibliography there). The results mentioned involve restricted sets of radii and large set of centers of spheres of integration. On the contrary, the problem under consideration in this paper involves a large set of radii (*e.g.,* $r > 0$) and a thin set of centers. This setup is natural in framework of the integral geometry since full spreads of spheres are used. The following result has been known a long time:

Theorem. *(F. John* [J]*, Courant-Hilbert* [CH p. 699 *ff.*]*). Let* Γ *be a hyperplane in* $\mathbb{R}^n$*. Suppose that* $f \in C(\mathbb{R}^n)$ *and the s.m.o.,* $Rf(x,r) = 0$ *for all* $(x,r) \in \Gamma \times \mathbb{R}_+$*. Then* f *is odd with respect to reflection around* Γ*.*

The simplest proof we know parallels the proof of Helgason's support theorem for the classical Radon transform on hyperplanes [He1]. We have a function that has zero integrals over balls of arbitrary radii with centers on Γ. Then successive differentiations along Γ, and successive application of Stokes' formula show that the function has an infinite number of vanishing power moments. These vanishing moments are equivalent to Γ-oddness of f.

This theorem says that $\ker_\Gamma R$, defined by (3), consists of all Γ-odd continuous functions. We want to describe $\ker_\Gamma R$ for an arbitrary set Γ. One can slightly generalize the theorem above by considering a finite union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$ of hyperplanes. Lin and Pinkus observed that in this case $\ker_\Gamma R$ consists of all functions in $C(\mathbb{R}^n)$ which are odd with respect to reflections around each hyperplane Γ_j and that is why $\ker_\Gamma R \neq 0$ if and only if the Coxeter group $W(\Gamma_1, \dots, \Gamma_n)$, generated by these reflections, is finite. In the following section we present a complete description of $\ker_\Gamma R$ when the operator R is defined on $C_c(\mathbb{R}^2)$. It appears that unions Γ of lines with finite Coxeter group are essentially the only sets for which $\ker_\Gamma R \neq 0$.

4. Injectivity conditions for the s.m.o. in the plane.

4.1. The result.

Theorem 4.1 [AQ1, AQ2]**.** *The following condition is necessary and sufficient for a set* $\Gamma \subset \mathbb{R}^2$ *to be a set of injectivity for the s.m.o., R, on the space* $C_c(\mathbb{R}^2)$*:*

$(*)$ *no translation* $t + \Gamma,\ t \in \mathbb{R}^2$*, is contained in a set of the form* $\Psi^{-1}(0) \cup F$*, where* Ψ *is a nonzero homogeneous harmonic polynomial and* F *is a finite set.*

Geometrically, the condition $(*)$ means that Γ is infinite and cannot be included in a system Σ_N of N lines having a common intersection point and equal angles π/N between any two closest lines, union with a finite set. The set Σ_N (Coxeter system of lines) has a finite reflection group.

4.2. Sketch of the proof.

Necessity. We can assume $t = 0$ in $(*)$. The space $\ker_{\Psi^{-1}(0)} R$ consists of all Σ_N-odd functions, $\Sigma_N = \Psi^{-1}(0)$. The Σ_N- skew symmetry of $f \in \ker_{\Psi^{-1}(0)} R$ gives f a sparse polar Fourier series. Being zero on the set F adds a finite number of additional conditions for the polar Fourier coefficients. This leads to a system of linear equation for these Fourier coefficients. Linear algebra arguments show that nontrivial solutions exist and therefore $\ker_{\Psi^{-1}(0)} R \neq 0$. It remains to note that $\ker_{\Psi^{-1}(0)} R \subset \ker_\Gamma R$.

Sufficiency is the difficult part of the proof. We have to describe the set $S[f]$, given by formula (4), for a fixed arbitrary function $f \in C_c(\mathbb{R}^2)$. Our aim is to prove that if $S[f]$ is infinite and $S[f] \neq \mathbb{R}^2$ ($i.e. f \not\equiv 0$), then there exists a point $t \in \mathbb{R}^2$, a nonzero harmonic homogeneous polynomial Ψ and a finite set F such that $t + S[f] = \Psi^{-1}(0) \cup F$. This proof consists of several steps.

Step 1. Algebraic Characterization of $S[f]$. Essentially, this step has been already described in 2.1. The representation (11) and the Bezout theorem imply that

$$S[f] = \Psi^{-1}(0) \cup F,$$

where Ψ is the greatest common divisor of all of the polynomials $Q_k[f], k = 0, 1, \dots$, and F is a finite set. Since the polynomial Q_k of the minimal nontrivial degree is harmonic (Proposition 2.1 and Corollary 2.2), the polynomial Ψ is a divisor of a harmonic polynomial.

Step 2. Geometric Analysis of $S[f]$.

Lemma 4.2. *Let $S = \Psi^{-1}(0)$ be an algebraic curve in the plane, where Ψ is a polynomial divisor of a nonzero harmonic polynomial. Then, only the two following cases are possible:*
(a) *$S = \Sigma_N$ (Ψ is homogeneous after some shift)*
(b) *S contains a ''hyperbola-like part" S_0 which is the union $S_0 = S_1 \cup S_2$ of two disjoint real-analytic curves having different asymptotic rays at infinity.*

The proof is based on an asymptotic analysis of the polynomial Ψ at the infinity and uses the Maximum Modulus Principle for harmonic polynomials.

Step 3. Microlocal Fourier Analysis and Support Theorem. Due to Lemma 4.2, it remains to prove that the condition (b) implies that S is a set of injectivity, *i.e.* $f \equiv 0$. To this end we use the calculus of real-analytic Fourier integral operators related to proofs in [BQ] [Q1, Q2]; the ideas are based on the microlocal methods of Guillemin [GS] in the real-analytic category [T, Ka]. We refer the reader to [AQ2] for details and will now only outline the main ideas.

In order to explain the background of the approach let us consider the linear Radon transform

$$Rf(\theta, c) = \int_{\langle \xi, \theta \rangle = c} f(\xi) dA(\xi) \quad \text{for} \quad f \in C_c(\mathbb{R}^n),\ (\theta, c) \in S^{n-1} \times \mathbb{R}.$$

In the case of s.m.o. the radius r is the analog of the parameter c and the center x is the analog of the direction θ. An open set of directions, θ is analogous to x lying on a non-affine surface S.

The standard proof of the fact that if $U \subset S^{n-1}$ is open then $Rf(\theta, c) = 0$ for $(\theta, c) \in U \times \mathbb{R}$ implies $f \equiv 0$, is based on the projection slice theorem:

$$\mathcal{F}^n f(r\theta) = \mathcal{F}^1_{c \to r}(Rf\,(\theta, c)),$$

where $\mathcal{F}^n$ and $\mathcal{F}^1$ are $n-$dimensional and $1-$dimensional Fourier transforms respectively. Since f has compact support, $\mathcal{F}^n f$ is real-analytic. Therefore, since $\mathcal{F}^n f(r\theta) = 0$ in the cone $C : r \geq 0,\ \theta \in U$, $\mathcal{F}^n f \equiv 0$ and so $f \equiv 0$.

This proof is obviously not applicable to the curved transform. In this case, we do not have a nice relation between the Fourier and Radon transforms because the fibration of $\mathbb{R}^n$ for spheres, (1b), does not consist of parallel hyperplanes. Nevertheless one can consider the projection formula for the linear Radon transform from a different point of view. This formula gives the expression:

$$Rf(\theta, c) = (\mathcal{F}^1)^{-1}_{r \to c} \mathcal{F}^n f(c\theta)$$

shows that R is a real-analytic elliptic Fourier integral operator associated to the Lagrangian manifold $N^*Z \setminus 0$ where $Z = \{(x, \theta, c) \mid \langle x, \theta \rangle = c\}$ and therefore the solution of the equation $Rf(\theta, c) = 0$, $(\theta, c) \in C$ must be real-analytic in wave front directions lying in the open cone C for all points in $\mathbb{R}^n$. Since f has compact support, this implies $f \equiv 0$ using a theorem of Hörmander, Kawai, and Kashiwara (discussed below) about analytic wave fronts at the boundary of supports.

This argument has some chance to be applicable, at least morally, to the curved transform because it does not impose strong requirements on the geometry of spreads. In fact the idea comes from the observation that the fibrations for curved Radon transforms are, infinitesimally, like that of hyperplanes–the global fibration in the linear case.

The first key point is the theorem [Hö, Theorem 8.5.1], which states that f has analytic wave front at conormal directions to the boundary of supp f. More precisely, if S is a smooth surface containing supp f and $x_0 \in \partial S \cap$ supp f, then $(x_0, \xi) \in \mathrm{WF_A}(f)$ when ξ is any normal vector to S at the point x_0.

The second key point in our proof is the fact that the s.m.o., R, is a real-analytic elliptic Fourier integral operator and the solution of the equation $Rf(x, r) = 0$, $(x, r) \in S \times \mathbb{R}_+$, must be "very smooth" (no analytic wave front at certain directions).

However, this smoothness does not come automatically. Analytic wave fronts of f can cancel at certain points. Analysis of the Lagrangian manifold of the s.m.o. shows that such cancellation can takes place at any pair of points x_0 and x_0^* (mirror points) which are symmetric with respect to the tangent line L_{a_0} to the curve Γ at the point a_0. Note that this is the analogous phenomenon on a infinitesimal level, to the fact that L_{a_0}- odd functions are in $\ker_{L_{a_0}} R$. Thus, we conclude that supp f must satisfy quite strong geometric symmetry conditions (in order for f to be non-analytic, it must have wave front at corresponding mirror points).

Now we assume that condition (b) from Lemma 4.1 holds. This implies that the curves S_1 and S_2 (branches of the "hyperbola") have two closest points $a_0 \in S_1, b_0 \in S_2$:

$$\text{dist }(S_1, S_2) = \text{ dist }(a_0, b_0).$$

The segment $[a_0, b_0]$ is perpendicular to the both tangent lines L_{a_0} and L_{b_0}, and these tangent lines are parallel to each other. Now, let $D(a_0)$ and $D(b_0)$ be the smallest disks centered at a_0 and b_0, respectively, that contain supp f. Finally, we show that the set supp f cannot satisfy the mirror condition above, with respect to both lines L_{a_0} and L_{b_0} and both disks $D(a_0)$ and $D(b_0)$, unless supp $f = \emptyset$ (since f is zero at the mirror points to the points on the boundaries of these disks that are in supp f). So, f is real-analytic in conormal directions to $\partial D(a_0)$ and $\partial D(b_0)$. So, by the first key point and the definitions of $D(a_0)$ and $D(b_0)$, $\partial D(a_0)$ and $\partial D(b_0)$ can't meet supp f. This contradiction completes the proof.

4.3. The result for the s.m.o. can be easily generalized to the transform over ellipses, by applying the corresponding affine transformation. Let $P(x_1, x_2) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}, a_1, a_2 > 0$, The ellipses are given by $P(\xi - x) = \frac{(\xi_1 - x_1)^2}{a_1^2} + \frac{(\xi_2 - x_2)^2}{a_2^2} = r^2$ and the transform is defined as in (1):

$$R_P f(x, \eta) = \int_0^{2\pi} f(x_1 + a_1 r \cos\theta, x_2 + a_2 r \sin\theta) d\theta, \; f \in C_c(\mathbb{R}^2),$$

where the measure $d\theta$ is the measure satisfying (1c) for $d\nu = a_1 a_2 r dr$.

Theorem 4.3. *R_P is injective on a set $\Gamma \subset \mathbb{R}^2$ if and only if no translation $t + \Gamma, t \in \mathbb{R}^2$, is contained in a set of the form $\Psi^{-1}(0) \cup F$, where Ψ is a nonzero homogeneous polynomial solution of the differential equation*

$$P^*(D)\Psi = \left(a_1^2 \frac{\partial^2}{\partial x_1^2} + a_2^2 \frac{\partial^2}{\partial x_2^2} \right) \Psi = 0$$

and F is a finite set.

The case of the Radon transform over hyperbolas $P(\xi - x) = r, P(\xi) = \frac{\xi_1^2}{a^2} - \frac{\xi_2^2}{b^2}, a, b > 0$, requires special arguments. For this case, we have only partial results so far.

5. Complete systems of radial functions in $C(\mathbb{R}^2)$

5.1. The results of §4 provide an answer to the question in [LP2], discussed in 1.1., for functions of two variables. Because of the equivalence stated in 1.1, Theorem 4.1 yields:

Theorem 5.1 ([AQ1, AQ2]). *Any function $f \in C(\mathbb{R}^2)$ can be approximated, uniformly on compact sets, by finite linear combinations of the form*

$$\sum g_i(|x - a_i|^2),$$

where $g_i \in C(\mathbb{R})$ and $a_i \in \Gamma \subset \mathbb{R}^2$, as long as the set Γ satisfies the condition $(*)$ of Theorem 4.1.

Thus, the space $\mathcal{L}(\Gamma) = \operatorname{span}\{g(|x-a|^2), g \in C(\mathbb{R}), a \in \Gamma\}$ is dense in $C(\mathbb{R}^2)$ if an essential part of Γ (the complement of a finite subset) is not a subset of some Coxeter system Σ_N. In this case $\ker_\Gamma R = \mathcal{L}(\Gamma)^\perp \cap C_C(\mathbb{R}^2) = \{0\}$.

Using the equivalence between (7a) and (7b) and (7c), we can rephrase Theorem 5.1.

Corollary 5.2. *Let $\mathcal{S} \subset \mathbb{R}$ be a set with an accumulation point in $\mathbb{R} \cup \{\infty\}$. Any function $f \in C(\mathbb{R}^2)$ can be approximated, uniformly on compact sets, by finite linear combinations of the form*

$$\sum c_i |x - a_i|^{2k_i},$$

where $c_i \in \mathbb{R}$, $a_i \in \Gamma \subset \mathbb{R}^2$, and $k_i \in \{0, 1, 2, \dots\}$ or by finite linear combinations of Gauss functions

$$\sum c_i \exp(\sigma_i |x - a_i|^2),$$

where $c_i \in \mathbb{R}$, $\sigma_i \in \mathcal{S}$ and $a_i \in \Gamma$, as long as the set Γ satisfies the condition $(*)$ of Theorem 4.1.

Corollary 5.2 justifies the following intriguing example. Let $\Gamma_1 = \{(n,n) \bigm| n \in \mathbb{Z}\}$. Then, the span of $\{\exp(\sigma|x-a|^2) \bigm| a \in \Gamma_1,\ \sigma \in \mathcal{S}\}$ is not dense in $C(\mathbb{R}^2)$ for any choice of $\mathcal{S}$. This is true since Γ_1 lies on a line or Coxeter set. Let $\Gamma_2 = \{([n/2], [(n+1)/2]) \bigm| n \in \mathbb{Z}\}$. Since Γ_2 is infinite and lies on no Coxeter system, if $\mathcal{S}$ has an accumulation point in $\mathbb{R} \cup \{\infty\}$, then the span of $\{\exp(\sigma|x-a|^2) \bigm| a \in \Gamma_2,\ \sigma \in \mathcal{S}\}$ is dense in $C(\mathbb{R}^2)$. For example, the system of Gauss functions $\{\exp(-m|x - a_n|^2), m \in \mathbb{Z}\}$ is complete in $C(\mathbb{R}^2)$ for $a_n = ([(n+1)/2], [n/2])$ and is not for $a_n = (n,n),\ n \in \mathbb{Z}$.

Since $\ker_{\Sigma_N} R$ consists of all Σ_N–odd functions in $C_c(\mathbb{R}^2)$ we have

Theorem 5.3 ([AQ1, AQ2])**.** *The closure of the space $\mathcal{L}(\Sigma_N)$ consists of all $f \in C(\mathbb{R}^2)$ having (r,θ) the sparse polar Fourier series in coordinates (r,θ):*

$$f(r,\theta) \sim \sum_{k=0}^{\infty} a_k(r) \cos k\theta + \sum_{k \notin N\cdot\mathbb{N}} b_k(r) \sin k\theta.$$

Definition. We say that a set $\Gamma \subset \mathbb{R}^2$ satisfies the *resonance condition* if there exist a finite subset $F \subset \Gamma$ and a point $t \in \mathbb{R}^2$ such that all points of $\Gamma \setminus F$ are visible from the point t under angles which are rational multiples of π.

This is, of course, a description of a shifted Coxeter system. Clearly, the condition $(*)$ of Theorem 4.1 means that the resonance condition does not hold for Γ.

Recall that L_a is the set of continuous radial functions centered at a. Theorem 5.1 states that the decomposition

$$C(\mathbb{R}^2) = \operatorname{cl}\left(\bigoplus_{a\in\Gamma} L_a\right)$$

into spherical waves centered at points of Γ holds if and only if Γ does not satisfy the resonance condition. Of course, $\Gamma = \Sigma_N$ satisfies the resonance condition; Σ_N has "eigenfrequences" and the direct sum $\oplus_{a\in\Sigma_N} L_a$ does not contain "eigenmodes" $\sin kN\theta$, θ is the angular coordinate in associated polar coordinates.

6. Applications to PDE

The following results are immediate corollaries of Theorem 4.1, due to equivalence discussed in 1.2.

Darboux equation.

Theorem 6.1. *Let u be a solution of the Cauchy problem (8) for the Darboux equation for $n = 2$, with Cauchy data $f \in C_c(\mathbb{R}^2)$ and $\Gamma \subset \mathbb{R}^2$. Then $u|_{\Gamma\times\mathbb{R}_+} = 0$ implies $u = 0$ as long as Γ satisfies the condition $(*)$ of Theorem 4.1.*

Heat equation.

Theorem 6.2. *Let u be a solution of the Cauchy problem (9) for the heat equation in the plane, with initial temperature distribution $f \in C_c(\mathbb{R}^2)$. Consider the zero temperature set $N[f] = \{x \in \mathbb{R}^2 \mid u(x,t) = 0 \text{ for all } t \in [0,T]\}$. Then the set $N[f]$ can be of the following three types:*

1. *$N[f] = \mathbb{R}^2 \quad (f \equiv 0)$,*
2. *$N[f]$ is a finite set,*
3. *$N[f] = \omega\Sigma_N \cup F$ for some $N \in \mathbb{N}$, where ω is a rigid motion of $\mathbb{R}^2$, Σ_N is a Coxeter system of straight lines and F is a finite set.*

In the case (3) the solution $u(x,t)$ is odd with respect to reflections around lines in $\omega\Sigma_N$, at any moment $t \in [0,T]$. So, if u has an infinite number of zeros for all time, then the initial temperature distribution must have this special skew symmetry. So, solutions to (8) that have compact support at $t = 0$ are uniquely determined by their values on any nonlinear smooth curve.

For the case when Γ is a curve, Theorem 6.2 can be formulated as solution of a free boundary problem.

Theorem 6.3. *Let Γ be a simple smooth curve that divides the plane into two domains Ω^+ and Ω^-.*

Consider the following free boundary problem for the heat equation with unknown boundary Γ:

$$\begin{gathered} u_t^{\pm} = c^2 \Delta u^{\pm} \quad \text{for} \quad (x,t) \in \Omega^{\pm} \times [0,T], \\ u^+|_{t=0} = u^-|_{t=0} \in C_c(\mathbb{R}^2), \\ u^{\pm}|_{\Gamma} = 0 \quad \text{for all} \quad t \in [0,T], \\ \frac{\partial u_+}{\partial \nu} = \frac{\partial u_-}{\partial \nu} \ \text{on } \Gamma, \end{gathered} \tag{12}$$

where ν is a unit normal vector field on Γ.

Then, nontrivial solutions are possible if and only if Γ is a straight line and u^+ and u^- are skew symmetric with respect to reflection around Γ.

Indeed, the pair (u^+, u^-) form a global solution vanishing on Γ. Theorem 6.2 states that, if this solution is nontrivial, then Γ must be a straight line.

Wave equation.

Theorem 6.4. *Nodal sets $N[f]$ of solutions of the Cauchy problem (10) for the wave equation in $\mathbb{R}^2$, with compactly supported initial velocity f, can be only of three following types:*

(1) $N[f] = \mathbb{R}^2$ *(the membrane does not oscillate at all),*
(2) $N[f]$ *is a finite set,*
(3) $N[f] = \omega\Sigma_N \cup F$ *for some $N \in \mathbb{N}$, where ω is a rigid motion of $\mathbb{R}^2$, Σ_N is a Coxeter system of straight lines and F is a finite set.*

Theorems 6.2 and 6.4 show that solutions with infinite sets of time-invariant zeros can occur only as result of strong symmetry of the initial data.

For instance, these sets cannot contain any small smooth nonlinear curve. For the heat equation it means that, if the initial temperature distribution has compact support, then it is impossible to have zero temperature along such a curve unless the temperature is identically zero.

In the case of the wave equation, we have obtained that, if the initial velocity is zero outside of a bounded region, then the oscillating membrane cannot remain stationary on a smooth curve that is not a segment of line.

Theorem 6.4 makes clear the character of oscillation of an infinite membrane with compactly supported initial velocity. There can be an infinite set of stationary points only if the initial velocity is skew symmetric with respect to some rigid motion of a Coxeter system $\omega\Sigma_N$; this skew symmetry persists in time.

Thus, only angular oscillations with respect to the polar coordinate system associated with Σ_N are possible.

Compact support for the initial velocity f is a crucial condition to our theorems. In general, for example, an infinite family of concentric circles can remain stationary. As we saw this is impossible when supp f is compact.

7. Applications to potential theory

7.1. Let μ be a regular Borel measure in $\mathbb{R}^n$. Define the Pompeiu transform of μ by

$$\tilde{R}\mu(x, r) = \mu(B(x, r)),$$

where $B(x, r) = \{\xi \in \mathbb{R}^n \bigm| |\xi - x| < r\}$. Of course, $\tilde{R}(x, r) = \int_{[0,r)} 1 dR\mu(a, \cdot)$ and, therefore, since μ is a regular Borel measure, $\tilde{R}\mu = 0$ iff $R\mu = 0$.

It was shown in 1.1 that for any $\Gamma \subset \mathbb{R}^n$ the subspace $\ker_\Gamma R \subset \ker{}_\Gamma \tilde{R}$ is dense in the weak topology in $C(\mathbb{R}^n)'$ and therefore the transforms R and $\tilde{R}$ have the same sets of injectivity. Then Theorem 4.1 implies

Theorem 7.1. *Let* $\mathcal{B} = \{B(x,r) \mid (x,r) \in \Gamma \times \mathbb{R}_+\}$ *be a family of disks in the plane. Any compactly supported measure* $\mu \in C(\mathbb{R}^2)'$ *can be identified by its values* $\mu(B),\ B \in \mathcal{B}$*, as long as the set* Γ *of centers satisfies the condition* $(*)$ *of Theorem 4.1.*

It is interesting to compare Theorem 7.1 with the existence of an infinite dimensional metric space in which all balls are not enough to identify measures [PT]. On the contrary, in the finite dimensional case, measures are determined by their values on sets of balls (of arbitrary radius) except for thin families of balls (disks) with special geometric symmetry.

7.2. Now we reformulate Theorem 7.1 in terms of Riesz potentials. For $\mu \in C(\mathbb{R}^2)'$ consider the Riesz potential

$$I_\lambda \mu(x) = \int_{\mathbb{R}^2} \frac{d\mu(\xi)}{|x-\xi|^\lambda}, \quad \lambda < 2.$$

If λ is fixed then μ is uniquely determined by values $I_\lambda \mu$ on any open subset $U \subset \mathbb{R}^2$ but μ is not determined if we know $I_\lambda \mu(x)$ for x on a curve.

On the other hand if $I_\lambda \mu(x) = 0$ for λ in an open interval, $\lambda \in (a,b)$ then $\tilde{R}\mu(x,r) = 0$ for all $r > 0$. The converse implication, obviously, is also true. Therefore Theorem 4.1 gives:

Theorem 7.2. *Let* $\mu \in C(\mathbb{R}^2)'$*. Consider a one-parameter family* $I_\lambda \mu,\ \lambda \in (a,b)$ *of Riesz potentials. If* $\Gamma \subset \mathbb{R}^2$ *and* $I_\lambda \mu|_\Gamma = 0$ *for* $\lambda \in (a,b)$ *then* $\mu = 0$ *provided* Γ *satisfies the condition* $(*)$ *of Theorem 4.1.*

8 A uniqueness theorem for the Laplace operator and nonlinear Fourier transform

8.1. Using the Fourier transform, one can formulate the result of §4 in another form which also may be interesting. First, we consider arbitrary dimension, n. The identity

$$Rf(x,r) = \int_{|y|=r} f(x+y)dA(y) = 0$$

is equivalent to the set of conditions:

$$\int_{\mathbb{R}^n} |y|^{2k} f(x+y)\ dy = 0, \quad k = 0, 1, \cdots$$

Applying the Fourier transform in the y-variable to these conditions yields:

$$\left(\Delta^k e^{i\langle\lambda,x\rangle} g(\lambda)\right)\Big|_{\lambda=0} = 0,\ g = \hat{f},\ k = 0, 1, \cdots \tag{13}$$

If $f \in \mathcal{D}'(\mathbb{R}^n)$, compactness of supp f is equivalent to g belonging to the Bernstein class $B(\mathbb{R}^n)$ of all real-analytic functions in $\mathbb{R}^n$ having a continuous extension to $\mathbb{C}^n$ that is an entire function of exponential growth.

Thus the problem of describing of set of injectivity of the s.m.o. has the dual form: find all sets $\Gamma \subset \mathbb{R}^n$ such that if (13) holds for all $x \in \Gamma$ then $g \equiv 0$.

The condition (13) can be rewritten in terms of the nonlinear Fourier transform (*cf.* [E]):

$$\mathcal{F}g(x,\alpha) = \int_{\mathbb{R}^n} e^{-\alpha|\lambda|^2 - i\langle\lambda,x\rangle} g(\lambda)\, d\lambda = 0 \tag{14}$$

for all $\alpha > 0$.

In these terms, the problem is to characterize all uniqueness sets for the transform $\mathcal{F}$ on the Bernstein class $B(\mathbb{R}^n)$, (*i.e.* the sets $\Gamma \subset \mathbb{R}^n$ for which $\mathcal{F}g(x,\alpha) = 0$ for $g \in B(\mathbb{R}^n)$ and $(x,\alpha) \in \Gamma \times \mathbb{R}_+$ implies $g = 0$. This question makes sense for other classes of functions g.

Theorem 4.1 answers this question for the case $n = 2$:

Theorem 8.1. *If $g \in B(\mathbb{R}^2)$ and equations (13)–(14) hold for $x \in \Gamma \subset \mathbb{R}^2$, then $g = 0$ provided Γ satisfies the condition $(*)$ of Theorem 4.1.*

8.2. The condition (14) can be rewritten in the equivalent form:

$$\int_{u \in SO(n)} e^{i\langle u\lambda, x\rangle} g(u\lambda)\, du = 0 \quad \forall\, x \in \Gamma,\ \lambda \in \mathbb{R}^n. \tag{15}$$

Here du is the Haar measure on the group $SO(n)$. Thus we deal with a uniqueness problem for the integral equation (15).

For $n = 2$ the equation (15) can be written in polar coordinates as

$$\int_0^{2\pi} e^{irs\cos(\varphi-\theta)} g(s,\theta)\, d\theta = 0,\ \forall s > 0,\ \forall (r,\varphi) \in \Gamma. \tag{16}$$

As is mentioned in §2, the generic case is when Γ is a curve. We can assume that $0 \in \Gamma$ and Γ is given in polar coordinates (φ, r) by equation $\varphi = \varphi(r), r > 0$. Theorem 4.1 states that the integral equation (16) has a unique solution $g = 0$ in $B(\mathbb{R}^2)$ unless $\varphi(r) = \varphi_0 =$ const. In this case all solutions satisfy $g(r, \varphi_0 - \varphi) = -g(r, \varphi_0 + \varphi)$.

It would be interesting to obtain the uniqueness theorem directly by investigating the integral equation (16). Stationary phase would seem to be an appropriate technique for this proof, but we did not succeed in proving the result in this way.

9. A Morera-type Theorem

Several modifications and variations of the Morera theorem are known that involve integration over circles (see *e.g.* the survey of Zalcman [Z3]). We suggest one more result of such type which is a simple consequence of Theorem 4.1 and Green's formula.

Theorem 9.1. *Let $f \in C^1(\mathbb{R}^2)$ be holomorphic in $\mathbb{C} \setminus K$ for some compact $K \subset \mathbb{C}$. Suppose that the complex integrals*

$$\int_{|w|=r} f(z+w)\, dw = 0$$

for all $r > 0$ and all $z \in \Gamma \subset \mathbb{C}$. Then f is an entire function as long as Γ satisfies the condition $(*)$ of Theorem 4.1.

Thus, if f is holomorphic outside of a compact set, then zero integrals over all circles centered on a smooth nonlinear curve imply f has no singularities on the compact K.

The proof of Theorem 9.1 consists of three steps. First, use Green's Theorem to reduce to an integral $\int_{|z-w|\le r} \frac{\partial f}{\partial \bar{z}} dA \equiv 0$ for $w \in \Gamma,\ r > 0$, and then differentiate with respect to r to get the integral $R\big(\frac{\partial f}{\partial \bar{z}}(w,r)\big) = 0$. Finally, use Theorem 4.1.

This Green's Theorem argument can be used along with a theorem of Volchkov [Q3] to show, for arbitrary $f \in C^1(\mathbb{C})$ that, if Γ consists of two concentric circles with well chosen radii and the integrals in Theorem 9.1 are zero, then $f \equiv 0$. An example using a Bessel's function shows that this is not true for arbitrary concentric circles.

10. Multidimensional case ($n > 2$)

In this section we present some partial results and a conjecture about the complete solution for the injectivity problem for the s.m.o. in $\mathbb{R}^n$ for $n > 2$.

Theorem 10.1. *(Necessary condition for sets of injectivity,* [AQ1, AQ2]*). Let $\Gamma \subset \mathbb{R}^n$ be a set of injectivity for s.m.o. on the space $C_c(\mathbb{R}^n)$. Then no translation $t + \Gamma, t \in \mathbb{R}^n$, is contained in a set of the form $\Psi^{-1}(0)$, where Ψ is a nonzero homogeneous harmonic polynomial.*

Proof. The proof is based on properties of the space of spherical harmonics. First, we can assume $t = 0$ and $\Gamma \subset \Psi^{-1}(0)$. Define the measure $\mu \in C(\mathbb{R}^n)'$ by

$$\int_{\mathbb{R}^n} g d\mu = \int_{\xi \in S^{n-1}} g(\xi)\Psi(\xi) dA(\xi), \quad \text{for} \quad g \in C(\mathbb{R}^n).$$

Let $e \in S^{n-1}$, let $SO(n-1, e) \subset SO(n)$ be the isotropy subgroup of the point e, and let $C_M(t)$ be the normalized Gegenbauer polynomial of the same degree, M, as Ψ (and order $\lambda = (n-3)/2$). Then,

$$\int_{k \in SO(n-1,e)} \Psi(k\xi) dk = \Psi(e) C_M(\xi \cdot e) \tag{17}$$

because Ψ is a homogeneous spherical harmonic and so the integral in (17) is a constant multiple of the unique normalized zonal spherical harmonic at $e \in S^{n-1}$, $C_M(\xi \cdot e)$.

Let $g \in C(\mathbb{R}^n)$ be a radial function and let $x_0 = re,\ e \in S^{n-1}$ be a point in Γ. Then, we can use (17) and the axial symmetry of $g(x_0 - y)$ about the ray $\overrightarrow{0e}$ to show $g * \mu(x_0) = 0$. Since $\mu \neq 0$ we conclude that $\ker_\Gamma \tilde{R} \neq 0$ where $\tilde{R}$ is defined in §7.2. Therefore, Γ is not a set of injectivity. $\square$

Theorems 4.1 and 10.1 provide motivation for the following conjecture.

Conjecture 10.2. *The following condition is necessary and sufficient for a set $\Gamma \subset \mathbb{R}^n$ to be a set of injectivity for the s.m.o. $R : C_c(\mathbb{R}^n) \to C(\Gamma \times \mathbb{R})$:*

(∗) *no translation $t + \Gamma, t \in \mathbb{R}^n$, is contained in a set of the form $\Psi^{-1}(0) \cup F$, where Ψ is a nonzero homogeneous harmonic polynomial and F is an algebraic variety of codim $F \geq 2$.*

The most difficult part in proving this conjecture is the sufficiency part, namely, the generalization of step 2 in the proof of Theorem 4.1 for the case $n > 2$. The reason is that not much is known about geometry of zero sets of harmonic polynomials of more that 2 variables. On the other hand, step 1 (structure of $S[f]$) and step 3 (support theorem) can be done for any dimension and therefore we are able to give some sufficient conditions for injectivity.

Definition. Let Γ be a hypersurface in $\mathbb{R}^n$. We call two points $a, b \in \Gamma, a \neq b$, *opposite* if a and b are points at which Γ is real-analytic, and the segment $[a, b]$ is perpendicular to the tangent spaces $T_a(\Gamma)$ and $T_b(\Gamma)$.

Theorem 10.3. *Any hypersurface $\Gamma \subset \mathbb{R}^n$, having at least two opposite points, is a set of injectivity for the s.m.o. R on $C_c(\mathbb{R}^n)$.*

<u>Examples.</u> The hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$ is a set of injectivity (opposite points $a = (-1, 0, 0), b = (1, 0, 0)$). The cone $x_1^2 + x_2^2 - 2x_3^2 = 0$ is the zero set of harmonic homogeneous polynomial and hence it is a set of non-injectivity due to Theorem 10.1.

At first glance, one might guess that if Ψ is polynomial divisor of a nonzero harmonic polynomial, then the algebraic variety $V = \Psi^{-1}(0)$ which is not a cone must contain opposite points. However, Michael Larsen observed that the polynomial in $\mathbb{R}^3$ $xyz - 1 = 0$ gives a counterexmaple to this guess. Nevertheless, sometimes the statement is true.

Theorem 10.4. *If the set $\Gamma \subset \mathbb{R}^n$ is axially symmetric and Γ is invariant with respect to rotations around a fixed straight line $l \subset \mathbb{R}^n$, then there are opposite points.*

Theorem 10.4 and results in §2 and §4, and Theorem 10.3, show that our Conjecture 10.2 is true in this case.

We can assume $l = \{x_1 = \cdots = x_{n-1} = 0\}$. Arguments given in §2 (see also §4, step 1 of the proof of Theorem 4.1) show that if Γ is not a set of injectivity then $\Gamma = \Gamma_0 \cup F$, where $\Gamma_0 = \Psi^{-1}(0)$, Ψ is polynomial divisor of a nonzero harmonic polynomial h in $\mathbb{R}^n$, and F is an algebraic variety of codim $F \geq 2$.

Now, we can apply averaging with respect to group $SO(n-1,e)$ of rotations around the line l, and therefore we can assume that the polynomial h has the form

$$h(x_1,\cdots x_n) = h_0(x_1^2+\cdots+x_{n-1}^2, x_n).$$

The polynomial $h_0(t,x_n)$ satisfies the differential equation

$$\left(4t\frac{\partial^2}{\partial t^2}+4\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x_n^2}\right)h_0=0. \tag{18}$$

Further analysis of the zero set of the polynomial h_0 in the plane (t,x_n) is similar to that in the proof of Theorem 4.1 (Lemma 4.2). We also use the fact that nonzero solutions of (18) cannot vanish on closed contours.

Remark. In the case of axial symmetry, considered in Theorem 10.4, sets of non-injectivity look as follows. These are subsets of unions $\Sigma_N \cup F$, where F is the "negligible" part–finite union of $(n-1)-$ spheres, generated by rotations around the line l of a finite set in the plane (t,x_n). The essential part Σ_N is union of N circular cones passing through zeros on the unit sphere S^{n-1} of the zonal spherical harmonic of degree N. More precisely, if l is the x_n-axes, then $\Sigma_N=\bigcup_{k=1}^N \mathcal{C}_K$, where $\mathcal{C}_K$ is the cone

$$\mathcal{C}_K=\{x_1^2+\cdots+x_{n-1}^2=\alpha_K^2x_n^2\}$$

and $\beta_K=\dfrac{1}{\sqrt{1+\alpha_K^2}}$ are zeros of the Gegenbauer polynomial $C_N(\beta_K)=0, \quad K=1,\cdots,N.$

11. The spherical mean operator on non-compactly supported functions

Now we consider the spherical mean operator $Rf(x,r)=\int_{S(x,r)}f(z)dA(z)$ with no restriction of boundedness for supports of functions in the domain of R.

The following result, pointed out to us by Larry Zalcman, gives a method of constructing of sets of non-injectivity for the operator R:

Proposition 11.1. *Let u be a solution of the Helmholtz equation in $\mathbb{R}^n$:*

$$\Delta u+\lambda^2u=0, \quad \lambda>0. \tag{19}$$

Then $Ru(x,r)=0$ for all $r>0$ as soon as $u(x)=0$. Therefore, if $u\not\equiv 0$ then $u^{-1}(0)$ is not a set of injectivity for the transform R.

Proof. Let u be a solution to the homogeneous Helmholtz equation and assume $u(x)=0$. We show $Ru(x,r)=0$, $\forall r>0$. Since the Helmholtz equation is elliptic and homogeneous, u is real analytic. Therefore, Weber's relation [We] or the generalized Pizzetti-Zalcman formula ([Z2] for arbitrary n) can be used to show for each $r>0$ that $Ru(x,r)$ is an infinite sum of terms including $u(x)$ and $\Delta^k u(x)$

for $k \in \mathbb{N}$ (see also [He2, Theorem 2.7, equation (25), p. 95] for homogeneous spaces). Because u satisfies the Helmholtz equation and $u(x) = 0$, this sum must be identically zero in r. $\square$

Larry Zalcman pointed out the radial example, $u(x) = J_{\frac{n}{2}-1}(\lambda|x|)/|x|^{\frac{n}{2}-1}$ where J_m is the Bessel function of order m. We obtain that any sphere $S(0, \alpha_K/\lambda)$ (where α_K is a zero of $J_{\frac{n}{2}-1}$) is a set of non-injectivity for the spherical mean operator.

This contrasts with the situation for compactly supported functions where any closed surface is a set of injectivity (Corollary 2.2).

In connection with Proposition 11.1 the following question arises:

Question. What are zero sets of real-analytic entire solutions of the Helmholtz equation (19)?

We do not know the answer for hypoelliptic algebraic curves in the plane, defined by equations

$$\gamma_{2p} : a_1 x_1^{2p} + a_2 x_2^{2p} = 1, \quad a_1, a_2 > 0.$$

The ellipses γ_2 are zero sets of nontrivial solutions of (19), but for γ_4 the question seems to be open. The following reduces the problem to uniqueness of an overdetermined boundary problem for a PDE. of elliptic type:

Proposition 11.2 [AS]**.** *The curve γ_{2k} is the zero set of a nonzero solution u to (19), which has Fourier transform (in the distribution sense) that is a measure in $\mathbb{R}^n$, if and only if the following boundary value problem has the unique solution $v = 0$:*

$$(a_1 \partial_{x_1}^{2k} + a_2 \partial_{x_2}^{2k})v = -v, \quad x_1^2 + x_2^2 \leq \lambda^2$$
$$\partial^\alpha v|_{x_1^2+x_2^2=\lambda^2} = 0 \ \ \textit{for} \ \ |\alpha| \leq k.$$

Note that the differential operator above has a discrete spectrum in the space of functions with boundary conditions: $\partial^\alpha v = 0$ for $|\alpha| \leq k-1$. The question is: do the extra boundary conditions for the $k-$th order derivatives imply that the spectrum is empty?

12. Open questions

We would like to conclude by list of open questions, some of them have been already mentioned above.

(1) Generalize Theorem 4.1 for any dimension (Conjecture 10.2).
(2) Describe sets of injectivity for the general transform R_P on compactly supported functions and measures.
(3) Describe sets of injectivity of the spherical mean operator (even for $n = 2$) defined on $C(\mathbb{R}^n), L^p(\mathbb{R}^n)$, other spaces that are larger than $C_c(\mathbb{R}^n)$.
(4) Obtain an analog of Theorem 5.1 in the category of entire functions in $\mathbb{C}^2$. (in $\mathbb{C}^n$).

(5) Describe the range of operator R_P and find the inverse operator in cases for which R_P is injective.
(6) The problem of decomposition into spherical waves, discussed in 1.1 and 1.2 can be generalized as follows.

Let $X = G/K$ be a homogeneous space of a Lie group G and T be a representation of G in a linear topological space F. Let π be an irreducible representation of K and T_π be the induced representation, $F_0 \subset F$ be the space of this representation, that is $F_0 = \{w \in F \mid T_k w = \pi(k)w, k \in K\}$. For any subset $\Gamma \subset G$ we consider the space

$$\mathcal{L}(\Gamma) = \operatorname{span}\{T_g w \mid w \in F_0, g \in \Gamma\}.$$

Describe all sets Γ such that $\mathcal{L}(\Gamma)$ is dense in F.

The problem discussed in this paper (§1) deals with the particular case: $G = M(n)$ - the group of rigid motions of $\mathbb{R}^n, K = SO(n), F = C(\mathbb{R}^n), T_g f(x) = f(gx)(g \in G), \pi$ is the trivial representation, Γ is a subset of parallel translations.

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