Electron Microscope Tomography over Curves

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(joint work with Hans Rullgård)

We define a general curvilinear Radon transform in $\mathbb{R}^3$, and we develop the microlocal properties of this transform. There are no inversion formulas for this transform, in general, and we give a local reconstruction method that recovers singularities of the object that are stably visible from the data. This is a type of regularization since we do not recover the function itself but singularities of the function that are stably reconstructed in a Sobolev sense. Our method is motivated by Lambda tomography, and it is a filtered back projection algorithm with a derivative filter. We characterize the singularities this algorithm reconstructs, and we show that some singularities are added to the reconstruction. Added singularities are inherent in any standard backprojection algorithm for this problem by the nature of the backprojection. Using our characterization of added singularities, we choose a derivative filter that will de-emphasize some of the added singularities. These results, their proofs, and reconstructions will appear in [3].

In *single object electron tomography (ET)*, images are taken of a single object over a finite number of rotations (called tilts) of the object in the electron beam. The standard model for single object ET assumes that electrons travel over lines and that the electron count at the detector is affected by the electrostatic potential f of the object. A more complete model will include the optics of the electron microscope, and information about the complete model is given in [1], as discussed in the talk of H. Kohr at this conference.

The theoretical work we describe here is motivated by practical work of Albert Lawrence, *et al.*, that shows when imaging larger objects using broader electron beams, the electrons farther from the center beam travel in helix-like curves. They have developed a reconstruction algorithm, TxBR [2], that uses gold markers in the projections to find the curves that the electrons travel over.

We now describe our general microlocal theory of a curvilinear Radon transform in $\mathbb{R}^3$. For each $\theta \in \,]a,b[$ (representing a tilt angle) and each $\boldsymbol{y} \in \mathbb{R}^2$ (representing a point on the detector plane for tilt θ), a smooth projection $\boldsymbol{p}_\theta : \mathbb{R}^3 \to \mathbb{R}^2$ defines the curves, which are given for $(\theta, \boldsymbol{y}) \in Y = \,]a,b[\, \times \mathbb{R}^2$ by

$$\gamma_{\theta,\boldsymbol{y}} := {\boldsymbol{p}_\theta}^{-1}(\{\boldsymbol{y}\}).$$

The *Curvilinear X-ray Transform* is given by

$$\mathcal{P}_{\boldsymbol{p}} f(\theta, \boldsymbol{y}) := \int_{\boldsymbol{x} \in \gamma_{\theta,\boldsymbol{y}}} f(x) ds.$$

The *backprojection operator* is given by

$$\mathcal{P}^*_{\boldsymbol{p}} g(\boldsymbol{x}) := \int_{\theta \in \,]a,b[} \varphi(\theta) g\left(\theta, \boldsymbol{p}_\theta(\boldsymbol{x})\right) d\theta$$

where φ is a cutoff function on $]a,b[$ that is equal to one on most of $]a,b[$ and is in $C_c^\infty(]a,b[)$. Since $\boldsymbol{x} \in \gamma_{\theta,\boldsymbol{p}_\theta(\boldsymbol{x})}$, $\mathcal{P}_{\boldsymbol{p}}^* g(\boldsymbol{x})$ is just an integral of g over all curves through $\boldsymbol{x}$.

Finally, our *singularity detection operator* is

$$\mathcal{L}(f) := \mathcal{P}_{\boldsymbol{p}}^* D \mathcal{P}_{\boldsymbol{p}} f$$

where D is a second order differential operator in $\boldsymbol{y}$ that is chosen to de-emphasize certain added singularities that we will describe below.

Clearly some conditions on the curves are necessary, and we will now describe our conditions and what they mean geometrically. We will let ∂_θ be the derivative in the theta direction, and $\partial_{\boldsymbol{x}}$ will be the gradient in $\boldsymbol{x}$.

(1) *For each $\theta \in]a,b[$, the curves $\gamma_{\theta,\boldsymbol{y}}$ are smooth, unbounded, and don't intersect.* Precisely, we assume that $(\boldsymbol{x},\theta) \mapsto \boldsymbol{p}_\theta(\boldsymbol{x}) \in \mathbb{R}^2$ is a C^∞ map. Fixing θ, $\boldsymbol{p}_\theta$ is a fiber map in $\boldsymbol{x}$ with fibers diffeomorphic to lines. This assumption will imply that $\partial_{\boldsymbol{x}} \boldsymbol{p}_\theta(\boldsymbol{x})$ has maximal rank (two).

(2) *The curves $\gamma_{\theta,\boldsymbol{y}}$ are different for different $(\theta,\boldsymbol{y}) \in Y$.*

(3) *Curves move differently at different points as θ changes.* The precise assumption is that for all $(\theta,\boldsymbol{y}) \in Y$ and for any two distinct points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ in $\gamma_{\theta,\boldsymbol{y}}$, the derivatives $\partial_\theta \boldsymbol{p}_\theta(\boldsymbol{x}_0)$ and $\partial_\theta \boldsymbol{p}_\theta(\boldsymbol{x}_1)$ are not equal.

(4) *The curves wiggle enough as θ changes.* Precisely, The 4×3 matrix $\begin{pmatrix} \partial_{\boldsymbol{x}} \boldsymbol{p}_\theta(\boldsymbol{x}) \\ \partial_\theta \partial_{\boldsymbol{x}} \boldsymbol{p}_\theta(\boldsymbol{x}) \end{pmatrix}$ has maximal rank (three). One can show this means that the normal plane to $\gamma_{\theta,\boldsymbol{y}}$ at $\boldsymbol{x} \in \gamma_{\theta,\boldsymbol{y}}$ changes as θ changes infinitesimally.

We now understand in an elementary way how our algorithm detects singularities. Let $\boldsymbol{x} \in \mathbb{R}^3$. Note that for each $\theta \in]a,b[$, $\boldsymbol{x} \in \gamma_{\theta,\boldsymbol{p}_\theta(\boldsymbol{x})}$. Therefore, the union of all curves in Y through $\boldsymbol{x}$ is

$$\Sigma_{\boldsymbol{x}} := \bigcup_{\theta \in]a,b[} \gamma_{\theta,\boldsymbol{p}_\theta(\boldsymbol{x})}$$

By assumption (3), $\Sigma_{\boldsymbol{x}}$ is smooth immersed surface except at $\boldsymbol{x}$, where it comes to a point [3].

For $\boldsymbol{x} \in \mathbb{R}^3$ and f a function of compact support $\mathcal{P}_{\boldsymbol{p}}^* \mathcal{P}_{\boldsymbol{p}} f(\boldsymbol{x})$ first integrates f over each curve through $\boldsymbol{x}$ and then averages over the curves through $\boldsymbol{x}$. Therefore,

$$\mathcal{P}_{\boldsymbol{p}}^* \mathcal{P}_{\boldsymbol{p}} f(\boldsymbol{x}) = \int_{\boldsymbol{z} \in \Sigma_{\boldsymbol{x}}} f(\boldsymbol{z})\, W(\boldsymbol{z},\boldsymbol{x})\, dA$$

where $W(\boldsymbol{z},\boldsymbol{x})$ is a smooth weight on $\Sigma_{\boldsymbol{x}} \setminus \{\boldsymbol{x}\}$. So, $\mathcal{P}_{\boldsymbol{p}}^* \mathcal{P}_{\boldsymbol{p}} f(\boldsymbol{x})$ is an integral of f over the surface $\Sigma_{\boldsymbol{x}}$. Since $\mathcal{L}$ is essentially $\mathcal{P}_{\boldsymbol{p}}^* \mathcal{P}_{\boldsymbol{p}}$ with a differential operator in the middle, $\mathcal{L} f$ detects singularities in essentially the same way as $\mathcal{P}_{\boldsymbol{p}}^* \mathcal{P}_{\boldsymbol{p}}$.

We use this idea to explain intuitively how $\mathcal{L}$ detects singularities. Let $\boldsymbol{x_0} \in \mathbb{R}^3$. If a singularity of f at $\boldsymbol{x_0}$ is conormal to some curve $\gamma_{\theta,\boldsymbol{p_\theta}(\boldsymbol{x})}$ then it should be detected by $\mathcal{P}_{\boldsymbol{p}} f$ [3]. To see this, let's do a thought experiment in which f is a characteristic function of a ball, B, and $\gamma_{\theta,\boldsymbol{p_\theta}(\boldsymbol{x_0})}$ is tangent to the ball at $\boldsymbol{x_0}$. Then, $\mathcal{P}_{\boldsymbol{p}} f$ will not be smooth near $(\theta, \boldsymbol{p}_\theta(\boldsymbol{x_0}))$ since $\mathcal{P}_{\boldsymbol{p}} f$ will go from 0 to nonzero as the curve moves in and out of the ball. Such singularities (which are called “visible”) will be detected by $\mathcal{L}$ (see [3] for a precise statement).

However, singularities from far away on $\Sigma_{\boldsymbol{x}_0}$ can also affect the reconstruction at $\boldsymbol{x_0}$. Imagine that $\Sigma_{\boldsymbol{x}_0}$ is tangent to the support of f, B at a point besides $\boldsymbol{x_0}$. Then, the integral $\mathcal{P}_{\boldsymbol{p}}^*\mathcal{P}_{\boldsymbol{p}}f$ will not be smooth at $\boldsymbol{x_0}$ since it will go from being zero to nonzero in a non-smooth way as $\boldsymbol{x}$ moves so that $\Sigma_{\boldsymbol{x}}$ moves in and out of the ball B. This adds singularities to the reconstruction at $\boldsymbol{x_0}$.

Precisely, in [3], we prove that $\mathcal{P}_{\boldsymbol{p}}$ is a Fourier integral operator associated with Lagrangian manifold $\Gamma = N^*Z \setminus 0$ where Z is the *incidence relation*

$$Z := \{(\theta, \boldsymbol{y}, \boldsymbol{x}) \in Y \times \mathbb{R}^3 \,\big|\, \boldsymbol{x} \in \gamma_{\theta,\boldsymbol{y}}\}$$

and N^*Z is its conormal bundle. Then $\mathcal{L}$ is a singular operator associated to the canonical relation $\Gamma^t \circ \Gamma$ and this relation above $\boldsymbol{x_0}$ is basically the visible directions (those conormal to curves through $\boldsymbol{x_0}$) $\times\ N^*(\Sigma_{\boldsymbol{x}_0}) \setminus 0$. So, singularities of f conormal at $\boldsymbol{x_0}$ to curves in the data set (visible singularities) will be detected by $\mathcal{L}f$ at $\boldsymbol{x_0}$, and singularities conormal to $\Sigma_{\boldsymbol{x}_0}$ at other points will be *added* to the reconstruction at $\boldsymbol{x_0}$. This is illustrated by our thought experiments above and is proven in [3].

In [3], we use our microlocal characterization of the added singularities to choose a differential operator D for $\mathcal{L}$ that will de-emphasize added singularities that are near to the reconstruction, and we show the algorithm works well to de-emphasize the added singularities.

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References

[1] D. Fanelli and O. Öktem. Electron tomography: A short overview with an emphasis on the absorption potential model for the forward problem. *Inverse Problems*, 24(1):013001, 2008.

[2] S. Phan, J. Bouwer, J. Lanman, M. Terada, and A. Lawrence. Non-Linear Bundle Adjustment for Electron Tomography. In *Proceedings CSIE09, March 31-April 1, 2009, Los Angeles, IEEE Computer Society*, 2009. to appear.

[3] E.T. Quinto and H. Rullgård, Local Singularity Reconstruction from Integrals over Curves in $\mathbb{R}^3$. preprint, 2010.