A Morera Theorem

for

spheres through a point in $\mathbf{C}^n$

Eric L. Grinberg and Eric Todd Quinto *

Department of Mathematics, Temple University, Philadelphia, PA 19122 USA
Current Address: Department of Applied Mathematics & Physics
Polytechnic University
6 MetroTech, Brooklyn, NY, 11201 USA
grinberg@math.temple.edu ; grinberg@magnus.poly.edu

Department of Mathematics, Tufts University, Medford, MA 02155 USA
equinto@math.tufts.edu

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Abstract: We prove a Morera theorem for the Radon transform integrating on spheres through a point in $\mathbf{C^n}$, and we give a counterexample when a smoothness hypothesis in the theorem fails. The theorem is proven by reduction to a support theorem for a Pompeiu transform. This theorem is proven using spherical functions and integral equations techniques as well as a support theorem for a generalized Radon transform on hyperplanes.

1. Introduction

The classical Morera Theorem states that, if $\int_C f(z)\,dz = 0$ for all simple closed curves in a region in the complex plane, then f is holomorphic in that region. Using harmonic and complex analysis, authors have proven more general Morera theorems that specify subclasses of curves which can be used to determine holomorphy in the plane (see *e.g.,* [BG 1986, BG 1988, BZ, Gl 1989, Gl 1990, Gl 1994, Za 1972, Za 1980]). Authors have generalized some of these results to $\mathbf{C}^n$ and other complex manifolds [Ag 1978, ABCP, Be, BZ, BG 1986, BG 1988, BP].

In [BZ], the authors prove the analogous theorem on non-compact rank one symmetric spaces. They also prove that if M is a compact rank-one symmetric space and $u \in L^1(M)$ and integrals of u over balls of one well chosen radius are zero, then $u = 0$, (Theorem 4 in [BZ]). As in $\mathbf{C^n}$, these results can give Morera theorems on other spaces. See [Za 1980, BCPZ, Za 1992] for excellent surveys of these problems.

We prove our Morera theorem by first proving a Pompeiu theorem for a related integral transform. Then, we use Stokes' Theorem to transfer the result to the desired Radon transform. This general framework has been used to good effect in the past. In [Za 1972] the author proves that if the integrals of a function over disks of two well chosen radii is zero, then the function is zero (see also [DL]). This theorem allows one to infer holomorphy of a function f if one knows that all integrals of f with respect to constant coefficient $(n, n-1)$ forms (see §2 for notation) are zero over all spheres of two well chosen radii. This Morera theorem follows by using Stokes' Theorem to reduce integrals of $f\,(*\mathbf{d\overline{z}}_j)$ over a sphere to Pompeiu integrals of $\frac{\partial f}{\partial \overline{z}_j}$ over a disk (see e.g., [Be]). Important local versions of this theorem are known and may be found in [BG 1986, BG 1988] and inversion methods are in [BGY].

The recent Ph.D. thesis of Yiying Zhou at Tufts University gives local support theorems for the sphere transform with two radii using microlocal techniques. Her proof was done for the ambient spaces $\mathbf{R^n}$, S^n, and $\mathbf{RP^n}$. Similar techniques can be used to prove two-sphere Morera theorems on real-analytic manifolds.

Morera theorems on various curves in the complex plane, including circles through the origin, were proven using microlocal analysis in [GlQ]. Morera theorems on complex manifolds were proven using related microlocal techniques in [GrQ].

Our results are closest in spirit to the work of Globevnik who proved lovely Morera theorems for various classes of circles and curves in the complex plane. Let D be a disk centered at the origin in the complex plane, and let f be a continuous function on D, In [Gl 1990], he proved that if f is C^∞ at the origin (see Definition 2.1), and the integral $\int f\,dz$ over all circles in D and surrounding the origin, is zero, then f is analytic in D. He also proved the analogous theorem for circles containing the origin [Gl 1989] and for rotation invariant symmetric curves containing the origin in [Gl 1994].

We reduce our Morera theorem to a Pompeiu theorem for disks with boundary containing the origin. We invert this transform using an integral equation close to the one solved in [CQ] and a support theorem for the generalized hyperplane Radon transform given [BQ].

2. The Morera Theorem and a Support Theorem

The following notation will be used throughout the article. Let $a \in \mathbf{C^n}$ (or $\mathbf{R^n}$). The sphere through the origin and with diameter the segment between 0 and a is denoted $\Gamma(a)$. Let $D(a)$ be the closed disk with boundary $\Gamma(a)$. Therefore, $a \in \operatorname{bd} D(a) = \Gamma(a)$.

Let $M \subset \mathbf{C^n}$ be open. Let $\Lambda^{2n-1}(M)$ be the set of differential forms of degree $2n-1$ on M with complex valued *real-analytic* coefficients and let $\Lambda^{(n,n-1)}(M) \subset \Lambda^{2n-1}(M)$ be the subset of $(n, n-1)$ forms (those $2n-1$ forms that are complex linear in the n *holomorphic* vector fields $\frac{\partial}{\partial z_j}$ and complex conjugate linear in the *anti-holomorphic* fields, $\frac{\partial}{\partial \overline{z}_j}$).

Recall that the *Hodge star* operator, $*$, on a manifold is defined in terms of an orientation on that manifold. On $\mathbf{C^n}$ we will choose the orientation $v = \mathbf{dx}_1 \wedge \mathbf{dy}_1 \wedge \cdots \wedge \mathbf{dx}_n \wedge \mathbf{dy}_n$ and define $*$ so that $\mathbf{d\overline{z}}_j \wedge *\mathbf{d\overline{z}}_j = v$. Let

$$L = \{ *\mathbf{d\overline{z}}_j \,\big|\, j = 1, \dots, n \} .$$

Then, the following property holds.

$$\begin{gathered} \text{For each } a \in \mathbf{C^n} \setminus 0, \text{ and each } w \in \Gamma(a), \\ \text{there is a form in } L \text{ that is nondegenerate over } T_w\Gamma(a). \end{gathered} \tag{2.1}$$

This is true because, restricted to the sphere $\Gamma(a)$, centered at $a/2$ and of radius $r = |a|/2$, the form $*\mathbf{d\overline{z}}_j$ is the function $(\overline{z}_j - \overline{a}_j/2)/r$ times the standard volume form on $\Gamma(a)$. Since L contains each $*\mathbf{d\overline{z}}_j$ for all $j = 1, \dots n$, L has a nondegenerate form at each point on each sphere $\Gamma(a)$. Of course, the linear span of L is the set of constant coefficient $(n, n-1)$ forms used in the theorems of [Be, BZ, BG 1986]. This nondegeneracy condition (2.1) is directly used in the microlocal proofs of Morera theorems in [GrQ]. It is the "moral" reason that the theorem is true: L includes enough forms so that all derivatives $\frac{\partial f}{\partial \overline{z}_j}$, $j = 1, \dots, n$ must vanish if all the integrals of $f\ (*\mathbf{d\overline{z}}_j)$ are zero.

Definition 2.1. *Let f be a function or distribution defined on an open set containing the origin. We say f is C^∞ at the origin if and only if for each positive integer k, there is a neighborhood of the origin U_k on which f has continuous derivatives up to order k.*

Theorem 2.1. *Let $\mathcal{A} \subset \mathbf{C^n}$ be open connected set containing the origin, and let $A = \cup_{a \in \mathcal{A}} D(a)$. Let $f \in C^1(A)$. Assume that f is C^∞ at the origin.*

Assume for all $a \in \mathcal{A}$ *and for* $j = 1, \ldots, n$ *that* $\int_{z \in \Gamma(a)} f\,(*\mathbf{d\bar{z}}_j) = 0$. *Then,* f *is holomorphic on* A.

This theorem is true if f is a distribution that is C^∞ at the origin in the sense of Definition 2.1. Theorem 2.1 as stated can be used to prove f is holomorphic in a neighborhood of the origin because f is a C^1 function near zero. Then, one can use a version of Lemma 3.1 that is valid for distributions to get f to be holomorphic in A.

The assumption that f is smooth at the origin is necessary as the following example shows. The proof is given in §3. Related examples exist for other sets of spheres. See [Gl 1990] in the plane, and, for area integrals in $\mathbf{R^n}$, [Jo], and Example 3.2 of [Q 1993].

Example 2.1. *Let* $k > 0$, $n > 0$ *and let* $m > k + n + 1$. *Let* $f : \mathbf{C^n} \to \mathbf{C}$ *be defined by* $f(z) \equiv f(z_1, \ldots, z_n) = z_1^m / \overline{z_1}^n$. *Then,* $f \in C^k(\mathbf{C^n})$ *and* f *has vanishing Morera integral over any sphere* S *that encloses or contains the origin in* $\mathbf{C^n}$ *with respect to each of the* $(n, n-1)$ *forms in* $L = \{*\mathbf{d\bar{z}}_j \mid j = 1, \ldots, n\}$.

The Morera Theorem 2.1 follows immediately from the following Pompeiu theorem. This is a support theorem for a type of integral transform on $\mathbf{R^n}$.

Theorem 2.2. *Let* $n \in \mathbf{N}$ *and let* $r > 0$. *Let* $\mathcal{A} \subset \mathbf{R^n}$ *be an open connected set containing the origin. Let* $A = \cup_{a \in \mathcal{A}} D(a)$ *and let* $f : A \to \mathbf{C}$ *be a continuous function. Suppose that* f *is* C^∞ *at the origin. Assume for all* $a \in \mathcal{A}$ *that* $\int_{z \in D(a)} f\,dV = 0$. *Then,* $f = 0$ *on* A.

The proof of the support theorem, Theorem 2.2, is given in §3. This immediately implies Theorem 2.1 by the following simple argument. Assume that $\int_{\Gamma(a)} f\,(*\mathbf{d\bar{z}}_j) = 0$ for $a \in \mathcal{A}$. Then, under the hypotheses of Theorem 2.1, Stokes' Theorem can be used to show $\int_{D(a)} \frac{\partial f}{\partial \bar{z}_j} dV = 0$ for all $a \in \mathcal{A}$. Theorem 2.2 implies that $\frac{\partial f}{\partial \bar{z}_j} = 0$ on A. As this is true for all j, f is holomorphic on A.

3. Proofs

Proof of Example 2.1. Note that up to a constant

$$d\left(f(z) \wedge *d\overline{z_i}\right) \equiv \begin{cases} z_1^m / \overline{z_1}^{n+1} \wedge (*1) & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

The form $*1$ is, by definition, $v = \mathbf{dx}_1 \wedge \mathbf{dy}_1 \wedge \cdots \wedge \mathbf{dx}_n \wedge \mathbf{dy}_n$, the orientation we have chosen on $\mathbf{C^n}$. Let D be the disk with center $a \in \mathbf{C^n}$ and of radius $r \geq |a|$ and let S be the boundary of D. S is, of course, a sphere containing or surrounding the origin. Using Stokes' Theorem on $\int_S f\,(*\mathbf{d\bar{z}}_j)$, one shows it is sufficient to prove the vanishing of the integral

$$I_1 = \int_D z_1^m / \overline{z_1}^{n+1}\,dV,$$

where dV is Lebesgue measure on $\mathbf{C^n}$.

We now project D onto the two-dimensional disk D' in the z_1 plane with radius r and center a_1, the first complex coordinate of a. By using a unitary rotation, we can assume a_1 is real. Using Fubini's theorem to calculate the integral, we get (up to a constant)

$$I_1 = \int_{z_1 \in D'} z_1^m / \overline{z_1}^{n+1} \left((r^2 - a_1^2) - z_1 \overline{z_1} + a_1 (z_1 + \overline{z_1}) \right)^{n-1} dA$$

where dA is the standard measure on the disk in $\mathbf{C}$. I_1 can be written as a sum of terms $I_{k,k'} = \int_{D'} z_1^k / \overline{z_1}^{k'} dA$ where $k - k' \geq m - (n+1)$.

By a result in [Gl 1990] one can show each $I_{k,k'}$ is zero. Let S' be the boundary of D'. Then, we can use Stokes' Theorem in the opposite direction on D' to write $I_{k,k'}$ as a sum of terms $\int_{S'} r^\ell e^{i\ell'\theta} dz$ where $0 < \ell < \ell'$, $\ell' > 0$ and ℓ has the same parity as ℓ'. Finally, since D' contains the origin in $\mathbf{C}$, we can use Theorem 2 (iii) in [Gl 1990] to see that each term is zero. □

Proof of Theorem 2.2. Let $\mathcal{A}$ be an open connected set in $\mathbf{R^n}$ containing the origin. Now, let $A = \cup_{a \in \mathcal{A}} D(a)$. Let $f \in C(A)$. Let $a \in \mathcal{A}$ and define

$$Rf(a) = \int_{x \in D(a)} f(x) dV \,. \tag{3.1}$$

Assume f is C^∞ at the origin and assume $Rf(a) = 0$ for all $a \in \mathcal{A}$. We will prove f is zero on A in two steps. First, we show that each spherical harmonic coefficient of f is zero in a disk centered at 0. Then, we use inversion in the unit sphere to reduce the final part of the problem to a support theorem for a real analytic Radon transform on hyperplanes in $\mathbf{R^n}$ to show $f = 0$ in A.

For $r > 0$, let $D(0, r)$ be the open disk centered at zero and of radius r. Let $r_0 > 0$ such that $D(0, r_0) \subset A$. We first show f is zero in $D(0, r_0)$.

Let $\ell \in \{0, 1, 2, \dots\}$ and let Y_ℓ be a spherical harmonic of degree ℓ [Se]. Equation (3.4) below shows that R maps spherical harmonics of degree ℓ to spherical harmonics of degree ℓ (that is, if $f_\ell(r)$ is a continuous function on $[0, r_0)$, then $R(f_\ell(r) Y_\ell(\tau))(s\theta)$ is a function of s times $Y_\ell(\theta)$). Let $r \geq 0$ and $\tau \in S^{n-1}$. We will show that if $f(r\tau) = f_\ell(r) Y_\ell(\tau)$ satisfies the hypotheses of Theorem 2.2, then f is zero in $D(0, r_0)$. Because an arbitrary f can be written as a sum of such terms, this will be sufficient to show arbitrary f is zero in $D(0, r_0)$.

We now calculate $R(f_\ell(r) Y_\ell(\tau))$, where Y_ℓ is any spherical harmonic of degree ℓ. The function f_ℓ can be extended to $(-r_0, r_0)$ to be an even function if ℓ is even and to be odd if ℓ is odd. Furthermore, by the assumption that f is C^∞ at the origin, there is an $r_1 \in (0, r_0)$ depending on ℓ such that

$$f_\ell(r) = \mathcal{O}(r^\ell) \text{ at } r = 0 \text{ and } f_\ell \in C^{(\ell+n-1)}\left((-r_1, r_1)\right) \tag{3.2}$$

We now show $f_\ell(r) = 0$ for $r \in (-r_1, r_1)$ by solving an integral equation for f_ℓ.

Let $s \in [0, r_0)$, and $\theta \in S^{n-1}$. We calculate $R(f_\ell Y_\ell)(s\theta)$ by writing the integral as an iterated integral over $D(s\theta)$. This gives an iterated integral for

$\tau \in S_\theta = \{\omega \in S^{n-1} \,\big|\, \omega \cdot \theta > 0\}$ and for $p \in [0, s\tau \cdot \theta]$.

$$R(f_\ell Y_\ell)(s\theta) = \int_{\tau \in S_\theta} Y_\ell(\tau) \int_{p=0}^{s\theta\cdot\tau} f_\ell(p) p^{n-1}\, dp\, d\tau \tag{3.3}$$

Here $\theta \cdot \tau$ is the real inner product of these vectors when viewed as elements of $\mathbf{R^n}$.

Now, let $C_\ell^\lambda(t)$ be the Gegenbauer Polynomial of degree ℓ; the Gegenbauer polynomials are orthogonal on $[-1,1]$ with weight $(1-t^2)^{\lambda-1/2}dt$. Next, we use the Funk-Hecke Theorem [Se] (really the uniqueness of spherical functions and the fact that $C_\ell^\lambda(\tau\cdot\theta)$ for $\lambda = (n-2)/2$ is a spherical function of degree ℓ centered at θ on S^{n-1}):

$$\begin{gathered} R(f_\ell Y_\ell)(s\theta) = Y_\ell(\theta) \frac{Vol(S^{n-2})}{C_\ell^\lambda(1)} \int_{t=0}^{1} F(st)\, C_\ell^\lambda(t)(1-t^2)^{\lambda-1/2}\, dt \\ \text{where } F(r) = \int_{p=0}^{r} f_\ell(p) p^{n-1} dp\,. \end{gathered} \tag{3.4}$$

By (3.2) and a simple change of variables in the expression for F in (3.4), we see

$$F(r) = \mathcal{O}(r^{(\ell+n)}) \text{ at } r=0 \text{ and } F \in C^{(\ell+n-1)}\big((-r_1, r_1)\big)\,. \tag{3.5}$$

To describe the inversion method, it will be easier to let $R_\ell f_\ell(s)$ be the coefficient of $Y_\ell(\theta)$ in (3.4). Substituting this identification into (3.4) and changing variable, we get

$$R_\ell(f_\ell)(s) = \frac{Vol(S^{n-2})}{s\, C_\ell^\lambda(1)} \int_{r=0}^{s} F(r)\, C_\ell^\lambda(r/s)\, (1-(r/s)^2)^{\lambda-1/2}\, dr\,. \tag{3.6}$$

Note that the right side of (3.6) is a function of f_ℓ as F is an integral of f_ℓ. Therefore, R_ℓ really is an operator on f_ℓ.

Equation (3.6) will be solved using the ideas in [CQ] and in particular equation (13) on p. 578 which says for $0 < r < p$:

$$\begin{aligned} &\int_r^p s^{(2\lambda-1)} C_\ell^\lambda(r/s) C_\ell^\lambda(p/s) \Big((1-(r/s)^2)((p/s)^2-1) \Big)^{\lambda-1/2} ds \\ &\qquad = \frac{\pi}{2^{n-3}} \left(\frac{\Gamma(\ell+2\lambda)}{\Gamma(\ell+1)\Gamma(\lambda)} \right)^2 \frac{(p-r)^{n-2}}{\Gamma(n-1)} \end{aligned} \tag{3.7}$$

We multiply (3.6) by $s^{2\lambda} C_\ell^\lambda(p/s)\big((p/s)^2-1\big)^{\lambda-1/2}$ and integrate from zero to p. One can use Fubini's theorem on the double integral since $C_\ell^\lambda(p/s)((p/s)^2 - 1)^{\lambda-1/2}$ grows like $(p/s)^{(\ell+n-3)}$ for $p > s$ and (3.5) holds and $s > r$. Next, we use (3.7) and finally, since F is sufficiently smooth, one can take $n-1$ derivatives of the resulting integral to get

$$F(p) = K \frac{d^{n-1}}{dp^{n-1}} \int_{s=0}^{p} R_\ell(f_\ell)(s)\, s^{2\lambda} C_\ell^\lambda(p/s)((p/s)^2-1)^{\lambda-1/2} ds \tag{3.8}$$

where K is a positive constant. If one takes one more derivative of F and divides by p^{n-1}, one gets an inversion formula for R_ℓ that recovers f_ℓ. This implies

$$f_\ell(r) = 0 \text{ for } r \in [0, r_1]\,. \tag{3.9}$$

Since f is not assumed to be smooth on all of $D(0,r_0)$, this proof does not show $f_\ell(r)$ is zero for $r > r_1$. The second part of the proof uses inversion in the unit sphere and then a support theorem for a real-analytic rotation invariant hyperplane transform to show f is zero on all of A.

Lemma 3.1. *Let $\mathcal{A} \subset \mathbf{R^n}$ be an open connected set containing the origin, and let $A = \cup_{a\in\mathcal{A}} D(a)$. Let f be a continuous function on A and assume f is zero in a neighborhood of the origin. Assume $Rf(a) = 0$ $\forall a \in \mathcal{A}$. Then, $f = 0$ on A.*

Assuming Lemma 3.1, the proof is completed as follows. Lemma 3.1 and (3.9) imply that $f_\ell(r)Y_\ell(\theta)$ is zero for $x = r\theta$ with $|x| < r_0$. As discussed above (3.2), since arbitrary f can be written as a sum of terms $f_\ell(r)Y_\ell(\theta)$ and R preserves spherical harmonics, any f satisfying the hypotheses of Theorem 2.2 will be zero on $D(0,r_0)$. Now, one can apply Lemma 3.1 to arbitrary f and $\mathcal{A}$ to show that f is zero in all of A. □

Proof of Lemma 3.1. Let $i : \mathbf{R^n}\setminus 0 \to \mathbf{R^n}\setminus 0$ be the inversion map in the unit sphere: $i(x) = x/|x|^2$. If $B \subset \mathbf{R^n}$ let $\tilde{B} = i(B)$. This inversion takes spheres $\Gamma(a)$ to hyperplanes and disks $D(a)$ to half spaces. We use the following notation for $a \in \mathbf{R^n}\setminus 0$

$$H(a/|a|, 1/|a|) = \{x \in \mathbf{R^n} \mid x\cdot a/|a| = 1/|a|\} \tag{3.10}$$

and for $f : \mathbf{R^n} \to \mathbf{C}$:

$$\tilde{f} = f \circ i.$$

It is a straightforward exercise to show that

$$\begin{aligned} Rf(a) &= \int_{p=1/|a|}^{\infty} \mathcal{R}\tilde{f}(a/|a|, p)dp \quad \text{where} \\ \mathcal{R}\tilde{f}(\omega, p) &= \int_{x\in H(\omega,p)} \tilde{f}(x)U(\left|x - p\omega\right|, p)dx_H \end{aligned} \tag{3.11}$$

where dx_H is the standard measure on the hyperplane and where $U(s,t)$ is a nowhere zero real-analytic function. The transform $\mathcal{R}$ is a rotation invariant Radon transform on hyperplanes. One proves this by writing $Rf(a)$ as an integral for a' on the line between 0 and a of an integral over $\Gamma(a')$. Then, one inverts in the unit sphere and gets an integral over parallel hyperplanes. Since the picture is rotation invariant, the weight U is of the form given in (3.11) (Proposition 2.2 in [Q 1983]). To show U is nowhere zero and real-analytic, one just writes the integral over $H(a/|a|,p)$ as an integral of $\tilde{f}$ over the hemisphere $S_{a'/|a'|}$ and compares to the integral of f over $\Gamma(a/(p|a|))$.

By taking a derivative with respect to p in the top integral of (3.11), we see

$$\mathcal{R}\tilde{f}(\omega,p) = 0 \text{ for } \omega/p \in \mathcal{A}. \tag{3.12}$$

Here we use that $\mathcal{A}$ is open in order to be able to take the derivative with respect to p: for each $a \in \mathcal{A}$, there is an $\epsilon > 0$ such that the segment $\{(1+r)a \,\big|\, |r| < \epsilon\}$ is in $\mathcal{A}$; one takes the derivative along this segment. The weight U for the Radon transform $\mathcal{R}$ is real-analytic and nowhere zero; $\{(\omega,p) \,\big|\, \omega/p \in \mathcal{A}\}$ is connected, open and unbounded; and f has compact support. Therefore, Theorem 2.1 of [BQ] immediately shows that $\tilde{f}$ is zero on $\cup_{a\in\mathcal{A}\setminus 0} H(a/|a|, 1/|a|)$. This implies f is zero on the inversion in the unit sphere of this set. That is: f is zero on A. □

Remark. *Ideas in the proof of Theorem 2.2 can be used to prove a generalization of the main theorem in [CQ] under the assumption f is continuous on A and C^∞ at the origin. That is, if $\int_{\Gamma(a)} f dx_\Gamma = 0$ $\forall a \in \mathcal{A}$ then $f = 0$ on A. The first part of the proof of Theorem 2.2 is replaced by (essentially) the proof given in [CQ] and the second part of the proof is replaced by a lemma like Lemma 3.1, but for the integral transform on $\Gamma(a)$, not the disk transform on $D(a)$.*

References

[Ag 1978] Agranovsky, M., *Fourier transform on $SL_2(\mathbf{R})$ and Morera type theorems*, Soviet Math. Dokl. **19** (1978), 1522–1526.

[ABC] Agranovsky, M., Berenstein, C.A., and Chang, D-C., *Morera Theorem for holomorphic H^p spaces in the Heisenberg group*, J. reine angew. Math. **443** (1993), 49–89.

[ABCP] Agranovsky, M., Berenstein, C.A., Chang, D-C., and Pascuas, D., *A Morera type theorem for L^2 functions in the Heisenberg group*, J. Analyse Math. **57** (1991), 282–296.

[Be] Berenstein, C.A., *A test for holomorphy in the unit ball of $\mathbf{C^n}$*, Proc. Amer. Math. Soc. **90** (1984), 88–90.

[BCPZ] Berenstein, C.A., Chang, D-C., Pascuas, D., and Zalcman, L., *Variations on the theorem of Morera*, Contemp. Math. **137** (1992), 63–78.

[BG 1986] Berenstein, C.A., and Gay, R., *A local version of the two circles theorem*, Israel J. Math. **55** (1986), 267–288.

[BG 1988] ——— *Le Problème de Pompeiu local*, J. Analyse Math. **52** (1988), 133–166.

[BGY] Berenstein, C.A., Gay R., and Yger, A., *Inversion of the local Pompeiu transform*, J. Analyse Math. **54** (1990), 259–287.

[BP] Berenstein, C.A., and Pascuas, D., *Morera and Mean-Value Type theorems in the Hyperbolic Disk*, Israel J. Math. **86** (1994) 61–106.

[BZ] Berenstein, C.A., and Zalcman, L., *Pompeiu's problem on symmetric spaces*, Comment. Math. Helv. **55** (1980), 593–621.

[BQ] Boman, J., and Quinto, E.T., *Support theorems for real analytic Radon transforms*, Duke Math. J. **55** (1987), 943–948.

[CQ] Cormack, A., and Quinto, E. T., *A Radon transform on spheres through the origin in* $\mathbf{R}^n$ *and applications to the Darboux equation*, Trans. Amer. Math. Soc. **260** (1980), 575–581.

[DL] Delsarte, J., and Lions, J.L., *Moyennes généralisées*, Comment. Math. Helv. **33** (1959), 59–69.

[Gl 1989] Globevnik, J., *Integrals over circles passing through the origin and a characterization of analytic functions*, J. Analyse Math. **52** (1989), 199–209.

[Gl 1990] ______ *Zero integrals on circles and characterizations of harmonic and analytic functions*, Trans. Amer. Math. Soc. **317** (1990), 313-330.

[Gl 1994] ______ *Holomorphic functions on rotation invariant families of curves passing through the origin*, J. Analyse Math. **63** (1994), 221–229.

[GlQ] Globevnik, J., and Quinto E., *Morera Theorems and microlocal Analysis*, J. Geometric Anal. **6** (1996), 19–30.

[GrQ] Grinberg, E.L., and Quinto E., *Morera Theorems for complex manifolds* `preprint`.

[Jo] John, F., *Plane Waves and Spherical Means* Interscience, New York (1966).

[Q 1983] Quinto, E.T., *The invertibility of rotation invariant Radon transforms*, J. Math. Anal. Appl. **91** (1983), 510–522; *Erratum*, J. Math. Anal. Appl. **94** (1983), 602–603.

[Q 1993] ______ *Pompeiu transforms on geodesic spheres in real analytic manifolds*, Israel J. Math. **84** (1993), 353–363.

[Se] Seeley, R., *Spherical Harmonics*, Amer. Math. Monthly **73** (1966), 115–121.

[Za 1972] Zalcman, L., *Analyticity and the Pompeiu problem*, Arch. Rat. Mech. Anal. **47** (1972), 237-254.

[Za 1980] ______ *Offbeat Integral Geometry*, Amer. Math. Monthly **87** (1980), 161–175.

[Za 1992] ______ *A bibliographic survey of the Pompeiu problem* in *Approximation of Solutions of Partial Differential Equations,* B. Fuglede, M. Goldstein, W. Haussmann, W. K. Hayman, and L. Rogge, Editors, Vol. 365, Series C: Mathematics and Physical Sciences, NATO ASI Series, Kluwer Academic, Boston, (1992), 185–194.