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Radon Transforms on Curves in the Plane

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ABSTRACT. We prove support theorems for Radon transforms integrating on curves in the plane. We consider transforms that integrate over translations of one fixed curve and transforms that integrate over circles of varying radius. The proofs require the microlocal analysis of the Radon transforms and a microlocal Holmgren theorem.

1. Introduction and Results

Radon transforms are integral transforms. A typical Radon transform integrates functions over members of a class of submanifolds. Radon transforms on curves in the plane are applied to fields as diverse as X-ray tomography [**Na**], geophysical imaging [**Mu**], and complex analysis. Our theorems are presented with the last application in mind. Recall that the classical Morera Theorem states that, if $\int_C f dz = 0$ for all closed curves in a region, then f is holomorphic in that region. More general Morera theorems specify subclasses of curves which can be used to determine holomorphy. In [**GQ**], support theorems for Radon transforms on curves are used to determine such subclasses of curves.

Support theorems for Radon transforms on curves in arc-length measure are known in many cases including lines [**He**], circles [**Za**], rotation invariant sets of curves [**Co, Ku**], and other cases [**Mu, Ro**]. Support theorems that are valid with more general measures are in [**BQ 1987, Q 1983, Q 1993**]. The main results of this article are general support theorems for fairly arbitrary sets of

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curves. We consider the case of translations of curves in 1.1 and the case of circles of varying radius in 1.2.

This research is based on the pioneering work of Guillemin and Sternberg [**Gu, GS**] that uses microlocal analysis to understand Radon transforms. The theory of real analytic Fourier integral operators is used to deduce analytic smoothness of a distribution f from support restrictions on $R_\mu f$ (Propositions 2.2 and 2.4). Then a theorem of Hörmander Kawai, and Kashiwara [**Hö, Ka**] about analytic singularities and support is used to deduce support restrictions on f from analytic smoothness of f (Lemma 3.2). These proofs follow naturally from the ideas in [**BQ 1987, Q 1993, Q 1980**].

We will now define the various Radon transforms and state our theorems. In §2, we will give the microlocal analysis of the transforms and in §3, we will prove the support theorems.

The Radon transforms in §1.1 and 1.2 are easily defined on domain $C_c(\mathbb{R}^2)$. However, our theorems are stated for distributions because we prove in §2 that these Radon transforms are Fourier integral operators. As shown in [**Tr**], Fourier integral operators are continuous operators on distributions.

1.1. Radon transforms on translates of curves. Let C be a real analytic curve in the plane. The set $Z = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x \in y + C\}$ is called the *incidence relation* for the Radon transform (1.1) below [**He**]. Now, let $\mu(x,y)$ be a nowhere zero real analytic function on Z. Let f be a continuous function of compact support, $f \in C_c(\mathbb{R}^2)$, then the *Radon transform* of f (associated to Z and μ) is defined for $y \in \mathbb{R}^2$ by

$$R_\mu f(y) = \int_{x \in y+C} f(x)\mu(x,y)ds(x) \tag{1.1}$$

where ds is the arc length measure on $y + C$. $R_\mu f(y)$ is simply the integral of f over the curve $y + C$ in weight μ. We exclude pathological cases in which the integral is not defined by requiring C to be either a smooth closed curve or an unbounded curve that does not oscillate too much. Our conditions will imply that C is a closed one-manifold without boundary that is imbedded in $\mathbb{R}^2$.

We include the weight μ in the definition of the Radon transforms above for several reasons. Allowing non-standard weights can help focus on properties intrinsic to the Radon transform rather than on specific symmetry properties that are valid only for special cases. This research is motivated by applications of these results to Morera theorems in complex analysis [**GQ**], and for this purpose, our theorems are needed for the non-standard measure dz.

We say that a smooth curve C is *flat to order one* at a point $w = (w_1, w_2) \in C$ if and only if the tangent line to C at w does not have higher than first order contact with C at w. If the curve is given by $x_2 = f(x_1)$ then the condition is, of course, equivalent to $f''(w_1) \neq 0$.

THEOREM 1.1. *Let C be a unbounded convex real analytic curve (that is, C divides the plane into two regions, one of which is convex). Assume C is flat to order one at all points on C. Let $\mathcal{A} \subset \mathbb{R}^2$ be open, connected, and non-empty. Let R_μ be the Radon transform on translates of C with nowhere zero real analytic weight μ. Assume $f \in \mathcal{E}'(\mathbb{R}^2)$ with $R_\mu f(y) = 0$ for all $y \in \mathcal{A}$ and assume, for some $y_0 \in \mathcal{A}$, the curve $y_0 + C$ is disjoint from* supp f. *Then for all $y \in \mathcal{A}$, $y + C$ is disjoint from* supp f.

The flatness assumption in Theorem 1.1 insures that C is strictly convex. The theorem is false if C is not strictly convex. The easiest example uses a line for C. Example 3.3 is more interesting. The hypothesis that there exists a $y_0 \in \mathcal{A}$ with $y_0 + C$ disjoint from supp f is also necessary, and an example is outlined after Example 3.3.

The next theorem deals with bounded curves.

THEOREM 1.2. *Let C be a smooth closed convex curve parameterized in polar coordinates by $r = r(\theta)$ where $r : [0, 2\pi] \to (0, \infty)$ is real analytic. Assume C is flat to order one at all points on C. Let $\mathcal{A} \subset \mathbb{R}^2$ be open, connected, and non-empty. Let R_μ be the Radon transform on translates of C with nowhere zero real analytic weight μ. Assume $f \in \mathcal{D}'(\mathbb{R}^2)$ with $R_\mu f(y) = 0$ for all $y \in \mathcal{A}$. Let D be the convex hull of C. Assume that, for some $y_0 \in \mathcal{A}$, the set $y_0 + D$ is disjoint from* supp f. *Then for all $y \in \mathcal{A}$, $y + D$ is disjoint from* supp f.

If one uses the classical measure, then R_1 is a convolution operator with a distribution in $\mathcal{E}'$ and so a simple Fourier transform argument proves this transform is injective on the domain of integrable functions. The proof of Theorem 1.2 is, morally, a microlocal version of this simple argument.

Negative results are known if the curve C is a circle. The example in [**Q 1993**] shows that the conclusion of Theorem 1.2 is false if one weakens the hypotheses about supp f to become: *for some $y_0 \in \mathcal{A}$, the curve $y_0 + C$ (not the set $y_0 + D$) is disjoint from* supp f. John has a counterexample if "$y_0 + D$" is replaced by "$y_0 + \text{int } D$" in this statement.

Positive results are also known if C is a circle. In this case and for $\mu \equiv 1$, this theorem follows from a result of Fritz John ([**Jo**, p. 115], see [**Q 1993**]). This theorem for transforms on circles with general measures is a special case of the theorem on Riemannian manifolds in [**Q 1993**]. In fact, the transform (with classical measures) on translates of two circles is injective on domain $C(\mathbb{R}^2)$, if their radii are 'well chosen' [**Za**] (see also [**BG**]).

1.2. Radon transforms on circles. The Radon transform integrating over all circles in the plane is overdetermined, since the set of all circles has dimension three, which is, of course, greater than the dimension of $\mathbb{R}^2$. Strong support theorems will be given in [**GQ**] for this general case. However, it is also appropriate to look for two dimensional subsets of circles on which uniqueness and support theorems hold. The set of translates of one fixed circle is two dimensional, and Theorem 1.2 includes this case. The two dimensional set we consider in this section is the set of all circles with centers constrained to lie on a fixed curve, γ. This is analogous to the admissible complex of lines in space meeting a fixed curve.

We will consider only the case that γ is a circle because the arguments become rapidly more complicated otherwise.

For $\theta \in [0, 2\pi]$, define $\overline{\theta} = (\cos\theta, \sin\theta)$. Let $t > 0$, then the circle of radius t centered at the origin is parameterized by $\gamma(\theta) = t\overline{\theta}$ for $\theta \in [0, 2\pi]$. This γ will be the curve containing the centers of the circles for the Radon transform (1.2). Let the weight $\mu(x, \theta, r)$ be continuous on the incidence relation for this Radon transform, $Z_t = \{(x, \theta, r) \in \mathbb{R}^2 \times [0, 2\pi] \times (0, \infty) \bigm| |x - t\overline{\theta}| = r\}$. Now we define the circular Radon transform for $f \in C(\mathbb{R}^2)$ as

$$S_t f(\theta, r) = \int_{\phi=0}^{2\pi} f(t\overline{\theta} + r\overline{\phi})\mu(t\overline{\theta} + r\overline{\phi}, \theta, r)d\phi. \tag{1.2}$$

This is the integral of f over the circle centered at $t\overline{\theta}$ of radius r in weight μ. We suppress the μ dependence for notational convenience; the radius, t of γ is important in the following theorem.

THEOREM 1.3. *Let $t > 0$ and choose $s \in (t, \infty)$. Let $R > \sqrt{s^2 - t^2}$, and let $\mathcal{A} = [0, 2\pi] \times (0, R)$. Let S_t and S_s be the Radon transforms defined in* (1.2) *with possibly different nowhere zero real analytic weights. Assume $f \in \mathcal{D}'(\mathbb{R}^2)$ with $S_t f(\theta, r) = S_s f(\theta, r) = 0$ for all $(\theta, r) \in \mathcal{A}$ and assume, for some $(\theta_0, r_0) \in \mathcal{A}$ with $r_0 > \sqrt{s^2 - t^2}$, the closed disk centered at $t\overline{\theta}_0$ of radius r_0 is disjoint from* supp f*. Then for each $(\theta, r) \in \mathcal{A}$, the disk centered at $t\overline{\theta}$ and of radius r and the disk centered at $s\overline{\theta}$ and of radius r are both disjoint from* supp f.

The proof of this theorem in §3 really gives a stronger theorem in which $\mathcal{A}$ is more general than $[0, 2\pi] \times (0, R)$. That stronger theorem is described in Remark 3.7

The 'shadow circle'–the circle of radius s–is not just an artifact of the proof. Example 3.8 shows that Theorem 1.3 is not true if integrals are known only over circles with centers on one circle. In requiring circles with centers on two curves ($|a| = t$ and $|a| = s$), Theorem 1.3 is a little like the theorem in [**Za**]; uniqueness follows in that theorem, if one has integrals over translations of all circles of *two* 'well chosen' radii. In Theorem 1.3, however, the radii of the curves of centers, $|a| = t$ and $|a| = s$, do not need to be 'well chosen.' Little is known about this case, and one might guess that a theorem that does not contradict Example 3.8 might be true without the 'shadow circle.'

Similar theorems are true in higher dimensions and the proofs are more complicated versions of the proofs given below.

Note added in proof: The author has just learned about the following recent results related to Theorem 1.3. These results are valid for the Radon transform that integrates functions over circles with centers on a given curve, γ, in arc length measure. Mark Agranovsky informed me of the proof of the following lovely theorem: if γ is not contained in the zero set of a non-zero harmonic polynomial, then this transform is injective on the domain $C_c(\mathbb{R}^2)$. For closed, real analytic curves (and some other real analytic curves) one can prove this using Proposition 2.4 and Lemma 3.2 below. The other results are for $f \in C(\mathbb{R}^2)$ and where the set of circle-centers, γ, is a collection of concentric circles. Valery Volchkov has recently proven a neat uniqueness theorem for this Radon transform when the circle-centers lie on two concentric circles, $|z| = s$, and $|z| = t$: if the radii, s and t, are "well chosen," and integrals of a function $f \in C(\mathbb{R}^2)$ over all circles centered on $|z| = s$, and $|z| = t$ are zero, then $f \equiv 0$. Our Theorem 1.3 requires an additional hypothesis compared to Volchkov's theorem: f is zero on a disk. However, our theorem doesn't require "well chosen" radii, and it is more "local" than Volchkov's theorem in that it can be used to conclude restrictions on supp f from restrictions on the support of its Radon transform. Larry Zalcman pointed out the intriguing counterexample of a function ($f(x) = J_0(|x|)$, the radial Bessel function) which has zero integrals over all circles with centers on an infinite collection of concentric circles (with "badly chosen" radii).

2. Radon transforms as analytic Fourier integral operators

The analytic wave front set, $\mathrm{WF_A}(f)$, of a distribution $f \in \mathcal{D}'(\mathbb{R}^2)$ is defined in **[Tr]** or **[Hö]**. When describing curves in polar coordinates, we assume that 0 and 2π are identified in $[0, 2\pi]$ and that functions of θ satisfy the natural compatibility conditions at 0 and 2π.

DEFINITION 2.2. Let C be a smooth curve in the plane and let $N^*C \subset T^*\mathbb{R}^2$ denote the conormal bundle of C. We say $x \in C$ and $x' \in C$ are *C-parallel* if and only if the tangent lines to C at the two points are parallel.

Now we can state the microlocal regularity theorem for the Radon transform (1.1).

PROPOSITION 2.2. *Let C be a curve that is parameterized in polar coordinates by $r = r(\theta)$ where $r : I \to (0, \infty)$ is real analytic on the interval $I \subset [0, 2\pi]$. Assume either that $r(\theta)$ goes to ∞ at each endpoint of I or that C is a closed curve. Let R_μ be the Radon transform on translates of C, (1.1), with nowhere zero real analytic weight μ. If C is unbounded, let $f \in \mathcal{E}'(\mathbb{R}^2)$, and if C is closed, let $f \in \mathcal{D}'(\mathbb{R}^2)$. Assume $R_\mu f$ is zero in an open neighborhood of $y \in \mathbb{R}^2$. Let $(x, \xi) \in N^*(y + C) \setminus 0$, and assume that f is zero in neighborhoods of all $(y + C)$-parallel points to x. Then $(x, \xi) \notin \mathrm{WF_A}(f)$.*

These growth conditions on C imply that C is a closed submanifold without boundary that is imbedded in $\mathbb{R}^2$. In general, Radon transforms detect only $\mathrm{WF_A}(f)$ conormal to the curve being integrated over. In this case, Proposition 2.2 shows that wavefront conormal to $y+C$ above x is detected as long as there are no singularities of f at the $(y+C)$-parallel points to x to mask this wavefront.

PROOF. The key to the proof of Proposition 2.2 is an understanding of the microlocal properties of the operator R_μ. This analysis is based on the double fibration [**GGS**] and the arguments of Guillemin and Sternberg [**Gu, GS**]. Recall that the incidence relation for this Radon transform is $Z = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x \in y+C\}$. The set N^*Z is the conormal bundle of Z in $T^*(X \times Y)$. The diagram needed for the proof is the microlocal analog of the double fibration:

$$\begin{array}{ccc} \Gamma = N^*(Z)\setminus 0 & \xrightarrow{\pi_2} & T^*(\mathbb{R}^2)\setminus 0 \\ \Big\downarrow{\scriptstyle \pi_1} & & \\ T^*(\mathbb{R}^2)\setminus 0 & & \end{array} \tag{2.1}$$

The maps are the natural projections onto the first or second factors.

To prove the statements about microlocal singularities in Proposition 2.2, we will show:

(2.2) covectors $(x,y;\xi,\eta) \in \Gamma$ and $(x',y;\xi',\eta) \in \Gamma$ have the same image under π_2 only if x and x' are $(y+C)$-parallel. π_2 is a local diffeomorphism.

First, we calculate N^*Z in good coordinates. For $w \in \mathbb{R}^2$ let $\arg w$ be the angle of w in polar coordinates (the angle between the positive horizontal axis and w). Although $\arg w$ is defined only $\mod 2\pi$, all relevant functions are 2π periodic. Points $(x,y) \in Z$ are determined by the equation $|x-y|^2 - r^2(\arg(x-y)) = 0$, and the differential of this equation gives a basis of the fibers of N^*Z. Coordinates on Z are given by:

$$\begin{aligned} I \times \mathbb{R}^2 &\to Z \\ (\phi, y) &\to (y + r(\phi)\overline{\phi}, y). \end{aligned} \tag{2.3}$$

Using these coordinates for Z gives coordinates for $N^*Z \setminus 0$ as follows:

$$\begin{aligned} &I \times \mathbb{R}^2 \times (\mathbb{R}\setminus 0) \to N^*Z \setminus 0 \\ &\qquad (\phi, y, a) \to \big(y + r(\phi)\overline{\phi}, y; a(r(\phi)\overline{\phi} - r'(\phi)\phi^\perp)(\mathbf{dx} - \mathbf{dy})\big). \end{aligned} \tag{2.4}$$

Here, $(w_1, w_2)\mathbf{dx} = w_1\mathbf{dx}_1 + w_2\mathbf{dx}_2$ is the covector in $T^*\mathbb{R}^2$ corresponding to $(w_1, w_2) \in \mathbb{R}^2$ and $\phi^\perp = (-\sin\phi, \cos\phi)$.

Equation (2.4) shows that π_1, and π_2 do not map to the zero section since $\overline{\phi}$ and $\phi^\perp$ are independent. So, R_μ is a Fourier integral operator associated to the Lagrangian manifold, Γ [**Tr,** Theorem 2.1, p. 316] (see also [**GS**] and [**Q 1980**]). This explains why R_μ can be evaluated on distributions. R_μ is elliptic since μ is nowhere zero.

The map π_2 is equivalent to the corresponding map in coordinates (2.4):

$$(\phi, y, a) \xrightarrow{\tilde{\pi}_2} (y; -a(r(\phi)\overline{\phi} - r'(\phi)\phi^{\perp})\mathbf{dy}). \tag{2.5}$$

First, note that the vector $(r(\phi)\overline{\phi} - r'(\phi)\phi^{\perp})$ in (2.5) is perpendicular to $y + C$ at $y + r(\phi)\overline{\phi}$. So, if (ϕ, y, a) and (ϕ', y, a') map to the same point under $\tilde{\pi}_2$, (2.5) shows that $y + r(\phi)\overline{\phi}$ and $y + r(\phi')\overline{\phi'}$ are $(y + C)$-parallel. This shows the first claim of (2.2).

To show that π_2 is an local diffeomorphism, one calculates the differential of $\tilde{\pi}_2$ and discovers that π_2 is a local diffeomorphism if and only if $\forall \phi \in I,\ r^2(\phi) - r(\phi)r''(\phi) + 2(r'(\phi))^2 \neq 0$. An elementary calculation shows this is equivalent to the assumption that C is flat to order one at all points on C.

Now, assume f is as in the hypotheses of Proposition 2.2. R_μ has been shown to be an analytic elliptic Fourier integral operator associated with Γ. The calculus of such operators implies the conclusion of Proposition 2.2. Let $(x, \xi) \in N^*(y + C) \setminus 0$ and assume f is zero near all $(y + C)$-parallel points to x. By (2.2), only singularities at $(y+C)$-parallel points to x can mask singularities of f above x. But, for all $(y+C)$-parallel points, x', to x, $\mathrm{WF_A}(f)$ is empty above x'. Therefore, singularities at $(y+C)$-parallel points to x cannot mask singularities at x. Since $R_\mu f$ is zero near y, $(x, \xi) \notin \mathrm{WF_A} f$. *(Precisely, we make a C^∞ partition of unity, $1 = \psi_p + \psi_x + \psi_0$, with the following conditions: ψ_p is one near each $y + C$-parallel point to x and ψ_p is sufficiently localized around the $(y + C)$-parallel points to x so that $\psi_p f = 0$; ψ_x is one near x and sufficiently localized around x so that $g \mapsto R_\mu \psi_x g$ satisfies the Bolker Assumption locally above (x, y) (the restricted π_2 is an injective immersion). Therefore, ψ_0 is zero near x and all its $(y + C)$-parallel points, so by (2.5) and the calculus of Fourier integral operators* [**SKK, Ka**], *$R_\mu \psi_0 f$ is smooth in directions (y, η) when $(x, y; \xi, -\eta) \in \Gamma$. Therefore, as $R_\mu f = R_\mu \psi_p f = 0$ and $(y, \eta) \notin R_\mu \psi_0 f$, $(y, \eta) \notin \mathrm{WF_A}(R_\mu \psi_x f)$. Since the operator $g \mapsto R_\mu \psi_x g$ satisfies the Bolker assumption above (x, y), $(x, \xi) \notin \mathrm{WF_A}(f)$.)* $\square$

The regularity theorem for the Radon transform (1.2) requires some smoothness conditions on f at certain points. Let

$$c_t(\theta, r) = \{x \in \mathbb{R}^2 \mid |x - t\overline{\theta}| = r\} \tag{2.6}$$

be the circle centered at $t\overline{\theta}$ and of radius r. This is the circle that $S_t f(\theta, r)$ integrates over.

Definition 2.3. Points x and x' in $c_t(\theta, r)$ are said to be $c_t(\theta, r)$-*mirror* if and only if they are reflections about the diameter of $c_t(\theta, r)$ that is perpendicular to $\overline{\theta}$.

The diameter in Definition 2.3 is the one tangent to the circle of radius t centered at the origin–the curve of centers of the circles $c_t(\theta, r)$.

PROPOSITION 2.4. *Let $t > 0$ and let S_t be the Radon transform in* (1.2) *with nowhere zero real analytic weight. Let $f \in \mathcal{D}'(\mathbb{R}^2)$. Assume $S_t f$ is zero in an open neighborhood of $(\theta, r) \in [0, 2\pi] \times (0, \infty)$. Let $(x, \xi) \in N^* c_t(\theta, r) \setminus 0$, and assume that f is zero in a neighborhood of the $c_t(\theta, r)$-mirror point to x. Then $(x, \xi) \notin \mathrm{WF_A}(f)$.*

If x is on the diameter of $c_t(\theta, r)$ perpendicular to $\overline{\theta}$, then x is its own mirror and Proposition 2.4 gives no useful conclusion about x.

PROOF. The proof will only be outlined because it is similar to the proof of Proposition 2.2. Recall that the incidence relation for S_t is $Z_t = \{(x, \theta, r) \in \mathbb{R}^2 \times [0, 2\pi] \times (0, \infty) \bigm| x \in c_t(\theta, r)\}$. The appropriate microlocal diagram is:

$$\begin{array}{ccc} \Gamma = N^*(Z) \setminus 0 & \xrightarrow{\pi_2} & T^*([0, 2\pi] \times (0, \infty)) \setminus 0 \\ \Big\downarrow \pi_1 & & \\ T^*(\mathbb{R}^2) \setminus 0 & & \end{array} \tag{2.7}$$

The goal is to prove that π_2 in (2.7) satisfies:

(2.8) covectors $(x, \theta, r; \xi, \eta) \in \Gamma$ and $(x', \theta, r; \xi', \eta) \in \Gamma$ have the same image under π_2 only if x and x' are $c_t(\theta, r)$-mirror. π_2 is a local diffeomorphism except above points (x, θ, r) where $x \in c_t(\theta, r)$ is its own mirror.

First, we calculate $N^* Z_t$ in good coordinates. Points $(x, \theta, r) \in Z_t$ are determined by the equation $|x - t\overline{\theta}|^2 - r^2 = 0$, and the differential of this equation gives a basis of the fibers of $N^* Z_t$. Coordinates for $N^* Z_t \setminus 0$ are:

$$\begin{aligned} &[0, 2\pi]^2 \times (0, \infty) \times (\mathbb{R} \setminus 0) \to N^* Z_t \setminus 0 \\ &\qquad (\phi, \theta, r, a) \to \big(x, \theta, r; a([x - t\overline{\theta}]\mathbf{dx} - [t(x - t\overline{\theta}) \cdot \theta^\perp]\mathbf{d}\boldsymbol{\theta} - r\mathbf{dr})\big) \\ &\qquad\qquad \text{where} \quad x = r\overline{\phi} + t\overline{\theta}. \end{aligned} \tag{2.9}$$

Equation (2.9) shows that π_1, and π_2 do not map to the zero section so, R_μ is a Fourier integral operator associated to the Lagrangian manifold, Γ [**Tr,** Theorem 2.1, p. 316] (see also [**GS**] and [**Q 1980**]). This explains why R_μ can be evaluated on distributions. R_μ is elliptic since μ is nowhere zero.

The map π_2 is equivalent to the corresponding map in coordinates (2.9):

$$(\phi, \theta, r, a) \xrightarrow{\tilde{\pi}_2} \big(\theta, r; -a([t(r\overline{\phi} \cdot \theta^\perp)]\mathbf{d}\boldsymbol{\theta} - r\mathbf{dr})\big). \tag{2.10}$$

Therefore, π_2 determines only $\overline{\phi} \cdot \theta^\perp$ so $x = t\overline{\theta} + r\overline{\phi}$ is known only up to its $c_t(\theta, r)$-mirror. This shows the first claim of (2.8). The calculation that $\tilde{\pi}_2$ is a local diffeomorphism except at self-mirror points is left to the reader, and the rest of the proof is parallel to that of Proposition 2.2. $\square$

3. Proofs

DEFINITION 3.1. Let $f \in \mathcal{D}'(\mathbb{R}^2)$ and let C' be a smooth curve. Let $x \in \operatorname{supp} f \cap C'$. We say that *supp f is on one side of C' near x* if and only if there is a neighborhood U of x such that C' divides U into two open sets and f is zero on one of these open sets.

PROOF OF THEOREM 1.1. The curve C is convex and flat to order one at each point, so C is strictly convex. The key observation is that, since C is strictly convex and unbounded, for each $x \in C$, there is no other C-parallel point. Therefore, Proposition 2.2 and the hypotheses of Theorem 1.1 imply that

$$\forall y \in \mathcal{A},\ \mathrm{WF_A}(f) \cap N^*(y + C) = \emptyset. \tag{3.1}$$

Since there is a $y_0 \in A$ such that $y_0 + C$ is disjoint from supp f and $\mathcal{A}$ is connected, there is a $y_1 \in \mathcal{A}$ such that $y_1 + C$ just touches supp f. Precisely, there is a point $x \in (\operatorname{supp} f) \cap (y_1 + C)$ such that supp f is on one side of $(y_1 + C)$ near x. Now, (3.1) implies that $\mathrm{WF_A}(f) \cap N^*(y_1 + C) = \emptyset$. However, the following theorem of Hörmander, Kawai, and Kashiwara [**Hö, Ka**] gives a contradiction.

LEMMA 3.2. *Let $h \in \mathcal{D}'(\mathbb{R}^2)$ and let C' be a smooth curve. Let $x \in \operatorname{supp} h \cap C'$ and assume* supp *h is on one side of C' near x. If $(x, \xi) \in N^*C' \setminus 0$, then $(x, \xi) \in \mathrm{WF_A}(f)$.*

This contradiction between (3.1) and Lemma 3.2 applied to $y_1 + C$ finishes the proof. $\square$

EXAMPLE 3.3. We construct a real analytic non-convex curve for which Theorem 1.1 is false by constructing a non-zero function in the null space of the Radon transform integrating over this curve in arc-length measure. Let C be the curve $x_2 = \sin x_1$ and let f be a function that is supported in $[0, 4\pi] \times [0, 10]$ and is $(+1)$ on $[0, 2\pi] \times [0, 10]$ and (-1) on $[2\pi, 4\pi] \times [0, 10]$. Then, the integral of f over any translate of C is zero because, for each $y \in \mathbb{R}^2$, the same length of $y + C$ meets $[0, 2\pi] \times [0, 10]$ as meets $[2\pi, 4\pi] \times [0, 10]$.

A counterexample to the conclusion of Theorem 1.1 can be made for an unbounded convex curve, C, when the hypothesis of Theorem 1.1 that some $y_0 + C$ is disjoint from supp f is left out. One chooses a convex curve, C, and constructs a function like the one in Example 3.3 that is both $(+1)$ and (-1) on strips that meet C. Then, one chooses a measure on each $y + C$ that changes as y varies around zero in such a way that the integrals on both parts of $(\operatorname{supp} f) \cap (y + C)$ cancel. For example, one can take a Gaussian weight on C and adjust the location of the maximum of the Gaussian for each y near zero so that the integrals over the strips cancel.

PROOF OF THEOREM 1.2. The proof is much like the one in [**Q 1993**] for Riemannian spheres so it will only be outlined. For each $\epsilon > 0$, let D_ϵ be the convex hull of $\cup_{|y|\le\epsilon}(y+C)$. Let $y_0 \in \mathcal{A}$ as in the statement of Theorem 1.2 and assume the conclusion of the theorem is false. Because $\mathcal{A}$ is open and connected, as in [**Q 1993**], one can find an $\epsilon > 0$ and a $y_1 \in A$ such that:

i) If $|y - y_1| \le \epsilon$, then $y \in \mathcal{A}$.
ii) int $(y_1 + D_\epsilon)$ is disjoint from supp f.
iii) $\partial(y_1 + D_\epsilon)$ meets supp f and $\partial(y_1 + D_\epsilon)$ is a smooth curve.

Assume $x \in \partial(y_1 + D_\epsilon) \cap (\text{supp } f)$ and let $(x, \xi) \in N^*\partial(y_1 + D_\epsilon) \setminus 0$. So, by, *ii*), *iii*), and Lemma 3.2, $(x, \xi) \in \mathrm{WF_A}(F)$. By *i*), there is a $y_2 \in \mathcal{A}$ such that $(y_2 + C) \subset (y_1 + D_\epsilon)$ and $x \in (y_2 + C)$. However, since C is strictly convex the one $(y_2 + C)$-parallel point to x is in int $(y_1 + D_\epsilon)$. Therefore, Proposition 2.2 implies that $(x, \xi) \notin \mathrm{WF_A}(f)$. This contradiction proves the theorem. □

PROOF OF THEOREM 1.3. We assume S_t, S_s, and f are as in the hypothesis of Theorem 1.3. The proof is fairly geometric and is easier to describe using some simple terminology (and for the reader to draw pictures illustrating the terms and the proof itself). Using a slight abuse of notation, we will refer to the curves of centers as $|a| = t$ and $|a| = s$. Recall that $c_t(\theta, r)$ is the circle centered at $t\overline{\theta}$ and of radius r, the circle of integration for $S_t f(\theta, r)$.

We define the *mirror diameter* of $c_t(\theta, r)$ to be the diameter tangent to $|a| = t$ at $t\overline{\theta}$. Therefore, mirror points (Definition 2.3) are reflections of each other in the mirror diameter of $c_t(\theta, r)$. Self-mirror points on $c_t(\theta, r)$ are the endpoints of the mirror diameter. The mirror diameter divides $c_t(\theta, r)$ into two open semicircles. We refer to the one closer to the origin as the *inner* open semicircle and the other one as the *outer* open semicircle.

The following lemma is one of the technical keys to the proof.

LEMMA 3.4. *Assume the distribution f is zero inside $c_t(\theta, r)$ and $S_t f$ is zero near (θ, r). Assume also that f is zero in a neighborhood of the inner (or the outer) open semicircle of $c_t(\theta, r)$. Then, f is zero in a neighborhood of the other open semicircle.*

Note that the proposition draws no inference at the self-mirror points.

PROOF. As $S_t f$ is zero near (θ, r) and f is zero in a neighborhood of the outer or inner open semicircle of $c_t(\theta, r)$, Proposition 2.2 implies that conormals to the other open semicircle are not in $\mathrm{WF_A} f$. Now, one uses Lemma 3.2 to conclude that f is zero on a neighborhood of the other open semicircle. □

In the proofs of Theorems 1.1 and 1.2, we used the curves that defined the Radon transform to eat away at supp f. For this theorem, we need a bigger set than $c_t(\theta, r)$. We use a capsule shaped object,

$$K(\theta, r) = \text{ Ch } \left(c_t(\theta, r) \cup c_s(\theta, r)\right) \tag{3.2}$$

where Ch A is the convex hull of the set $A \subset \mathbb{R}^2$.

The following lemma will be needed as we eat away at supp f.

LEMMA 3.5. *Let f and S_t, S_s be as in the hypotheses of Theorem 1.3 and let $r_3 > \sqrt{s^2 - t^2}$. Assume $S_t f = S_s f = 0$ in a neighborhood of (θ_3, r_3). Assume $S_s f(\theta, r) = 0$ in a neighborhood of each (θ, p) such that $c_s(\theta, r)$ is tangent to $K(\theta_3, r_3)$ and is contained in $K(\theta_3, r_3)$. Assume f is zero on* int $K(\theta_3, r_3)$. *Then, f is zero on a neighborhood of $K(\theta_3, r_3)$.*

PROOF. We can use Lemma 3.4 to show $f = 0$ in neighborhoods of the outer open semicircle of $c_s(\theta_3, r_3)$ and the inner open semicircle of $c_t(\theta_3, r_3)$. This can be done because the opposite semicircles are inside $K(\theta_3, r_3)$.

Let ℓ be one of the closed line segments on $\partial K(\theta_3, r_3)$. We show that f is zero in a neighborhood of ℓ. Let x_3 be the point of intersection of ℓ and $c_s(\theta_3, r_3)$. For each point $x \in \ell \setminus \{x_3\}$, there is a circle $C = c_s(\theta_1, r_1)$ that has center on $|a| = s$, that is tangent to ℓ, and that is contained in $K(\theta_3, r_3)$. By the hypothesis of this lemma, $S_s f$ is zero near (θ_1, r_1). The point of tangency of C with ℓ is the only point on C that could possibly meet supp f. That point is not a self-mirror point since $x \neq x_3$ so $C \neq c_s(\theta_3, r_3)$. Therefore, Lemma 3.4 can be used to show f is zero in a neighborhood of this point. (*This argument works for points $x \in \ell \setminus \{x_3\}$ that are outside of $|a| = s$ because, for each such x, the circle C exists.*)

Since $r_3 > \sqrt{s^2 - t^2}$, the above argument is valid for all points on $\ell \setminus \{x_3\}$.

Now we show f is zero in a neighborhood of x_3. We will do this by constructing circles centered on $|a| = s$ that have only a part of their outer semicircles outside of $K(\theta_3, r_3)$. We will use these to eat away at supp f at x_3. The circles we use are easier to describe in local coordinates. Choose local coordinates $\overline{x} = (\overline{x}_1, \overline{x}_2)$ so that $s\overline{\theta}$ is the origin and $x_3 = (r_3, 0)$. Let the circle $|a| = s$ correspond in local coordinates to the circle γ_s of radius s and centered at $(0, -s)$. Let $\delta \in (0, s - t)$, and let C_δ be the smaller circle centered on γ_s that goes through $(r_3, -\delta)$ and $(r_3, 0)$. The center of C_δ is $\overline{a} = (\sqrt{s\delta - \delta^2/4}, -\delta/2)$.

Let r_δ be the radius of C_δ. Using estimates, one can show that for sufficiently small δ, the slope of the mirror diameter of C_δ is more negative than the slope of the segment between $\overline{a}$ and $(r_3, -\delta)$. Therefore, the mirror diameters and inner open semicircles of C_δ and all circles centered at $\overline{a}$ of radius less than r_δ are contained in $K(\theta_3, r_3)$. If δ is sufficiently small so that $S_s f$ is zero near these circles, Lemma 3.4 can be used to show that supp f is disjoint from all circles centered at $\overline{a}$ of radius less than or equal to r_δ. Thus, $x_3 \notin$supp f. (*This argument to show $x_3 \notin$ supp f works for all $r_3 > 0$.*)

This finishes the proof that supp f is disjoint from all of $K(\theta_3, r_3)$. $\square$

Now we have the tools to prove the theorem. We start with $c_t(\theta_0, r_0)$ and its inside disjoint from supp f. We first show that $K(\theta_0, r_0)$ is disjoint from supp f.

LEMMA 3.6. *Under the hypotheses of Theorem 1.3, if* $c_t(\theta_0, r_0)$ *and its inside is disjoint from* supp f, *then* $K(\theta_0, r_0)$ *is disjoint from* supp f.

PROOF. First, note that the outer half circle of $c_t(\theta_0, r_0)$ is outside of $|a| = s$ as $r_0 > \sqrt{s^2 - t^2}$. There is a radius $r_1 > 0$ such that the self-mirror points on the circle $c_s(\theta_0, r_1)$ are on $c_t(\theta_0, r_0)$ and only the outer semicircle of this circle is outside $c_t(\theta_0, r_0)$. Therefore, we can use Lemma 3.4 to show that $c_s(\theta_0, r_1)$ is disjoint from supp f. (*If not, there is a largest radius* $r_2 \leq r_1$ *such that* supp f *meets* $c_s(\theta_0, r_1)$ *and is locally on one side of* $c_s(\theta_0, r_1)$. *Apply Lemma 3.4 to this circle to show* supp f *is disjoint from this circle and its interior. This contradiction shows that* $c_s(\theta_0, r_1)$ *and its inside is disjoint from* supp f.)

Let $D_t(\theta_0, r_0)$ be the closed disk that has boundary $c_t(\theta_0, r_0)$. We have just proven that $D_t(\theta_0, r_0) \cup K(\theta_0, r_1)$ is disjoint from supp f. Assume $K(\theta_0, r_0)$ is not disjoint from supp f. As we let r_1 increase to r_0, there is a first radius, r_2 such that $D_t(\theta_0, r_0) \cup K(\theta_0, r_2)$ meets supp f, and supp f is disjoint from the interior of $D_t(\theta_0, r_0) \cup K(\theta_0, r_2)$. By Lemma 3.4, supp f is disjoint from the outer open semicircle of $c_t(\theta_0, r_2)$. The boundary of $D_t(\theta_0, r_0) \cup K(\theta_0, r_2)$ is made up of semicircles and two line segments. The segments are outside of $|a| = s$ because the outer semicircle of $c_t(\theta_3, r_3)$ is outside of $|a| = s$. Since these segments are outside of $|a| = s$, the arguments in the proof of Lemma 3.5 that showed the segment ℓ is disjoint from supp f can be used here (see the italicized statements in the proof of Lemma 3.5). This shows that the entire set $D_t(\theta_0, r_0) \cup K(\theta_0, r_2)$ is disjoint from supp f, and so $r_2 = r_0$ and the lemma is proved. □

These lemmas are now used to finish the proof. By Lemma 3.6, we find that $K(\theta_0, r_0)$ is disjoint from supp f. As in [**BQ 1987**] we can find a path in $\mathcal{A}$ from (θ_0, r_0) to some (θ_3, r_3) such that $r_3 > \sqrt{s^2 - t^2}$ and $\partial K(\theta_3, r_3)$ meets supp f but int $K(\theta_3, r_3)$ does not meet supp f. Now, we can use Lemma 3.5 to draw a contradiction. This finishes the proof of the theorem. □

REMARK 3.7. Note that the proof is valid if $\mathcal{A} \subset [0, 2\pi] \times (0, \infty)$ is an open connected set such that if $(\theta_3, r_3) \in \mathcal{A}$ then so are $(\theta_3, \sqrt{s^2 - t^2})$ as well as all (θ, r) such that $c_s(\theta, r)$ is tangent to $\partial K(\theta_3, r_3)$ and $c_s(\theta, r) \subset K(\theta_3, r_3)$. This requirement is all that is needed for one to apply Lemmas 3.5 and 3.6.

EXAMPLE 3.8. Let S_2 be the Radon transform that integrates over circles centered on $|a| = 2$ in standard arc length measure. We construct a smooth non-zero function $F(x)$ with $S_2 F \equiv 0$ and with $\{x \in \mathbb{R}^2 \mid |x| \leq 1 \text{ or } |x| = 3\} \subset$ supp F but supp f is disjoint from $\{x \in \mathbb{R}^2 \mid 1 < |x| < 3\}$.

A similar construction, outlined below, shows that there is no general Morera theorem for this set of circles. The construction provides a function that is not holomorphic but that has zero integrals with respect to the analytic measure, dz, over all circles centered on the curve $|a| = 2$.

This example shows that Theorem 1.3 is not valid without a shadow curve.

If $s \in (2, \infty)$ and $S_2F \equiv 0 \equiv S_sF$, then Theorem 1.3 implies that $F \equiv 0$. As noted above, $\{x \in \mathbb{R}^2 \mid |x| = 1 \text{ or } |x| = 3\} \subset \operatorname{supp} f$. The reason this example is possible is that wavefront at the c_2 mirror points on $|x| = 1$ and $|x| = 3$ cancel.

CONSTRUCTION. Let $k : [0, \infty) \to \mathbb{R}$ be continuous. Then, for $r \in [0, \infty)$ we define Radon transform $S_2k(r)$ to be $S_2[k(|x|)](\theta, r)$ for any $\theta \in [0, 2\pi]$. This is well defined independent of θ because $k(|x|)$ is a radial function and S_2 is rotation invariant since the weight $\mu \equiv 1$.

To define F, we first specify a smooth, non-negative, even function $f_1 \in C^\infty(\mathbb{R})$ with $\operatorname{supp} f_1 = [-1, 1]$. Next we use f_1 to construct the smooth function $\overline{f}_2$ with $\operatorname{supp} \overline{f}_2 \subset [3, 7)$ by solving an integral equation so that $-S_2f_1(r) = S_2\overline{f}_2(r)$ for $r \in [0, 5)$. Then $F(x) = f_1(|x|) + \overline{f}_2(|x|)$ will satisfy the conditions in the example for $|x| < 7$ and $r \in [0, 5)$. Finally we will outline how to extend the construction.

Now, $\overline{f}_2(u)$ is constructed. By the choice of f_1,

$$S_2f_1(r) \text{ is smooth and } \operatorname{supp} S_2f_1(r) = [1, 3]. \tag{3.3}$$

If $f(u)$ is a function supported in $[3, 7)$ then an exercise using the law of cosines shows for $w = r + 2 \in [3, 7)$ that

$$S_2f(w-2) = 4\int_3^w \frac{u(w-2)f(u)du}{\sqrt{w-u}\sqrt{(w+u)(u^2-(w-4)^2)}}. \tag{3.4}$$

Because of (3.4), the equation

$$-S_2f_1(w-2) = S_2\overline{f}_2(w-2) \tag{3.5}$$

is a first kind Abel integral equation for $\overline{f}_2(u)$. By (3.3), the function $S_2f_1(w-2)$ is zero to infinite order at $w = 3$. Therefore, the integral equation (3.5) satisfies the hypotheses of Theorem B [**Q 1983**]. (see also [**Yo**]). So, one can solve (3.5) for the function $\overline{f}_2(u)$ for $u \in [3, 7)$. As S_2f_1 is smooth, $\overline{f}_2$ is smooth on $[3, 7)$ [**Yo**]. As $r = 1 \in \operatorname{supp}(S_2f_1)$, $u = 3 \in \operatorname{supp} \overline{f}_2$. Since $S_2f_1(w-2)$ is zero to infinite order at $w = 3$, $\overline{f}_2(u)$ is zero to infinite order at $u = 3$; so, $\overline{f}_2$ can be smoothly extended to $[0, 7)$ to be zero on $[0, 3]$ and with $3 \in \operatorname{supp} \overline{f}_2$. Now, by (3.5),

$$S_2\big(f_1 + \overline{f}_2\big)(w-2) = 0 \text{ for } w \in [2, 7). \tag{3.6}$$

The extension is completely analogous to the argument in Example 3.2 of [**Q 1993**] so it will only be outlined. First let $\epsilon \in (0, 1/4)$. Define $f_2(u)$ to be equal to $\overline{f}_2(u)$ for $u \in [0, 7-\epsilon/2)$ and to smoothly taper to zero and have support in $[3, 7)$. Now, solve the integral equation equivalent to (3.4), $-S_2(f_1+f_2)(w-2) = S_2\overline{f}_3(w-2)$ on $w \in [7-\epsilon/2, 11-\epsilon/2)$, for $\overline{f}_3(u)$. By the choice of f_1 and f_2, $\overline{f}_3$ is smooth, supported in $[7-\epsilon/2, 11-\epsilon/2)$. One defines f_3 by cutting off $\overline{f}_3$

smoothly between $11 - 3\epsilon/4$ and $11 - \epsilon/2$. One continues this process as in **[Q 1993]**.

If one integrates with respect to the analytic measure, dz, then $S_2 f(\theta, r)$ is no longer radial, and the integral (3.4) changes in unessential ways. $S_2 f(\theta, r)$ becomes $ire^{i\theta}$ times an integral like (3.4) but with the addition of $\left[(w-2)-(w^2-u^2)/4\right]/(w-2)$ to the inside of the integral. The rest of the proof is essentially the same because the new integral equation is one to which Theorem B **[Q 1983]** applies. $\square$

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