RADON TRANSFORMS SATISFYING
THE BOLKER ASSUMPTION

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ABSTRACT. Using the techniques of real-analytic microlocal analysis, we prove support theorems for Radon transforms integrating with non-zero real-analytic measures. We assume the transforms satisfy the Bolker Assumption, a microlocal injectivity condition that simplifies the microlocal analysis.

1. INTRODUCTION.

Radon transforms are integral transforms. Given a real-analytic Riemannian manifold $X$ and a collection, $Y$, of submanifolds of $X$, one can define a Radon transform $R_\mu$ that integrates functions on $X$ over each submanifold in $Y$ with respect to a measure $\mu$. Further structure on $X$, $Y$—the double fibration (2.1) [GGS, He 1984]—is needed to prove injectivity and other properties of this transform. Guillemin and Sternberg (e.g., [GS]) have used this structure and microlocal analysis to understand properties of Radon transforms (see also [GU]). These advances make it natural to investigate general Radon transforms with arbitrary measures.

A support theorem for a Radon transform gives restrictions on the support of a function from information on the support of its Radon transform. Classical techniques have been used to prove support theorems for the transforms with canonical measures on hyperplanes [He 1964, GGV] and horocycles [He 1973]. The problem is more subtle for transforms with general measures, and support theorems have been proven under either strong symmetry conditions such as rotation invariance (e.g., [Q 1983]) or strong smoothness conditions on the measure such as real-analyticity (e.g., [BQ 1987]). A counterexample [Bo 1993] of a non-injective Radon transform on lines with a positive $C^\infty$ measure shows the importance of strong hypotheses on the Radon transform for injectivity to hold.

In this article, we give a general support theorem, Theorem 2.2, that is valid for all Radon transforms with real-analytic measures that satisfy the Bolker Assumption (Definition 2.1). We also give a local support theorem, Proposition 2.3. In §2, the main definitions and theorems are given. The microlocal properties of $R_\mu$ are recalled and the main theorem are proven in §3.

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2. Definitions and main results.

Let $X$ and $Y$ be connected real-analytic manifolds. Let $Z$ be a connected imbedded submanifold of $X \times Y$. The double fibration (below left) [GGS] has a corresponding diagram on the cotangent spaces (below right):

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi_Y} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_X} \\
X & \xrightarrow{\pi_Y} & T^*(X) \setminus 0
\end{array}
\Gamma = N^*(Z) \setminus 0 \xrightarrow{\pi_Y} T^*(Y) \setminus 0
\tag{2.1}
$$

where $\pi_X$ and $\pi_Y$ are the natural projections in each case. In (2.1), $N^*Z$ is the conormal bundle of $Z$ in $T^*(X \times Y)$. If $A$ is a submanifold of $X$, then we will let $N^*A$ denote the conormal bundle of $A$ in $T^*X$.

When the left diagram in (2.1) is a double fibration, the natural projections $\pi_X$ and $\pi_Y$ are fibrations with connected fibers and $\pi_X$ is proper [GGS, He 1984]. The double fibration defines submanifolds of $X$, $\hat{y} = \pi_X \circ \pi_Y^{-1}(y)$, all of which are diffeomorphic to the fiber of $\pi_Y$. Let the weight $\mu(x,y)$ be real-analytic and nowhere zero on $Z$. For $f \in C_c(X)$, the Radon transform of $f$ is defined at $y \in Y$ by

$$
R_\mu f(y) = \int_{x \in \hat{y}} f(x)\mu(x,y)dm_y(x)
\tag{1.2}
$$

where $dm_y$ is the measure on $\hat{y}$ induced from the Riemannian structure on $X$. By duality, $R_\mu : \mathcal{E}'(X) \rightarrow \mathcal{E}'(Y)$ is continuous. The key definition is:

**Definition 2.1.** The double fibration (2.1) (or the Radon transform $R_\mu$) satisfies the Bolker Assumption if and only if $\pi_Y : \Gamma \rightarrow T^*Y \setminus 0$ is an injective immersion.

This is named after Ethan Bolker who formulated an analogous definition for finite Radon transforms. Our main theorem is:

**Theorem 2.2.** Let the Radon transform $R_\mu$ be defined by a real-analytic double fibration that satisfies the Bolker assumption. Assume the weight $\mu$ is nowhere zero and real-analytic. Let $A$ be an open connected subset of $Y$. Assume $f \in C_c(X)$ with $R_\mu f(y) = 0$ for all $y \in A$ and assume, for some $y_0 \in A$, $f$ is zero to infinite order on $\hat{y}_0$. Then, for all $y \in A$, $\hat{y}$ is disjoint from supp $f$.

This theorem implies the “standard” theorem of [BQ 1987, Q 1993a,c]: if $f$ is zero in a neighborhood of $\hat{y}_0$ and the other hypotheses of Theorem 2.2 hold, then $f$ is zero on all $\hat{y}$ for $y \in A$. This weaker theorem holds for $f \in \mathcal{E}'(X)$ as can be seen from the proof in §3. The key to Theorem 2.2 which is not in these previous theorems is a microlocal regularity theorem of Boman [Bo 1992]. Boman has used this theorem to prove support theorems for hyperplane transforms and $\mathcal{S}(\mathbb{R}^n)$ [Bo 1991].

Many classical Radon transforms satisfy the Bolker assumption. Among those are the transforms in Euclidean space integrating over: $k$-planes, real, complex, and quaternionic hyperplanes, and the Cayley line, [e.g., GS, Q 1993c]; the point horocycle transform on non-compact rank one symmetric spaces [Gu, Q 1993a]; and the maximal totally geodesic transforms on compact two point symmetric spaces (by

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$^1$That is, for each $n \in \mathbb{N}$, $f(x)$ is bounded by a fixed multiple of the $n$th power of the geodesic distance between $x$ and $\hat{y}_0$.}
the projection argument in [Q 1983, p. 520]). Support theorems have been proven for classical measures on these spaces [e.g., He 1964, GGV, Gl, He 1973] and with non-standard measures [BQ 1987 (real hyperplanes), Q 1993c (other planes), Q 1993a (rank one horocycle transform), and Q 1983, Q 1987 (transforms on compact two point homogeneous spaces)]. If the Bolker Assumption does not hold for a Radon transform, then more subtle geometric arguments (e.g., [BQ 1993, GQ]) are needed to prove support theorems.

An example in [HSSW] demonstrates the necessity of some growth assumption on \( f \) near some \( y_0 \in A \). Using local versions of our arguments, one can easily prove local support theorems in the spirit of those in [Gl]. One such theorem (proved in §3) is:

**Proposition 2.3.** Let the Radon transform \( R_\mu \) be defined by a real-analytic double fibration that satisfies the Bolker assumption. Assume the weight \( \mu \) is nowhere zero and real-analytic. Let \( x_0 \in X \) and let \( U \) be an open connected neighborhood of \( x_0 \). Let \( f \in C_c(X) \) and assume \( R_\mu f(y) = 0 \) for all \( y \) that meet \( U \). Assume that \( f \) is zero to infinite order at \( x_0 \). Then, \( f \) is zero in \( U \).

Note that the only growth restriction on \( f \) away from \( x_0 \) is that \( f \) has compact support−\( U \) does not necessarily contain supp \( f \). If more is known about the geometry of the manifolds \( \hat{y} \), then local hole theorems such as Theorem 4.3 of [Q 1993a] are valid.

3. Proofs.

Let \( \text{WF}_A(f) \) denote the analytic wave front set of \( f \in \mathcal{D}'(X) \) [Hö 1983]. The two keys to the proofs of the main theorem are: Proposition 3.1, the microlocal properties of \( R_\mu \); and Lemma 3.2, a theorem of Hörmander, Kawai, and Kashiwara [Hö 1983] on the analytic wave front set of a distribution at the boundary of its support.

**Proposition 3.1.** Let \( R_\mu \) be a Radon transform with nowhere zero real-analytic weight \( \mu \) that satisfies the Bolker Assumption. Let \( f \in \mathcal{E}'(X) \) and let \( y_1 \in Y \). Assume \( R_\mu f(y) = 0 \) for all \( y \) in an open neighborhood of \( y_1 \). Let \( (x, \xi) \in N^*\hat{y}_1 \setminus 0 \). Then \( (x, \xi) \notin \text{WF}_A(f) \).

**Proof of Proposition 3.1.** It is well known that if a Radon transform satisfies the Bolker Assumption, then \( R_\mu \) is an elliptic Fourier integral operator associated with the Lagrangian manifold \( \Gamma \) and \( R_\mu^*R_\mu \) is an elliptic pseudodifferential operator [e.g., GS, Q 1980]. The statements of Proposition 3.1 follow immediately from the calculus of analytic elliptic Fourier integral operators [SKK], [Ka]: As \( R_\mu f(y) = 0 \) for \( y \) near \( y_1 \), then \( (\text{WF}_A R_\mu f) \cap (T_{y_1}^*Y \setminus 0) = \emptyset \). Using the calculus of elliptic Fourier integral operators one sees that \( \pi_X \circ \pi_Y^{-1}(T_{y_1}^*Y \setminus 0) \cap \text{WF}_A f = \emptyset \). But \( \pi_X^{-1}(T_{y_1}^*Y \setminus 0) = N^*\hat{y}_1 \setminus 0 \) (see e.g., [Q 1993a, Proposition 3.2]).

The next lemma allows us to eat away at supp \( f \) using hypersurfaces. It is a special case of Theorem 8.6 [Hö 1983].

**Lemma 3.2** [Hö 1983]. Let \( S \subset X \) be a \( C^2 \) hypersurface. Let \( x \in S \). Let \( f \in \mathcal{E}'(X) \) and assume \( f \) is zero on one side of \( S \) locally near \( x \) (i.e., there is a neighborhood \( U \) of \( x \) such that \( U \setminus S \) consists of two connected open sets, \( x \) is in the boundary of each, and \( f \) is zero on one of these sets). If \( x \in \text{supp} f \) and \( (x, \xi) \in N^*(\partial S) \setminus 0 \), then \( (x, \xi) \in \text{WF}_A(f) \).

**Proof of Theorem 2.2.** By the assumptions of the theorem and the conclusion of Proposition 3.1, we see that \( N^*\hat{y}_0 \cap \text{WF}_A(f) = \emptyset \). Therefore, as a distribution,
$f$ can be restricted to $\hat{y}_0$. And, because $f$ is zero to infinite order along $\hat{y}_0$, $f$ is flat along $\hat{y}_0$ as a distribution (the proof uses the definition in [Bo 1992] and Proposition 2.5.11 of [Hö 1971] to show all distribution derivatives of $f$ are zero along $\hat{y}_0$). Under these hypotheses ($N^*\hat{y}_0 \cap \text{WF}(f) = \emptyset$ and $f$ is flat along $\hat{y}_0$), the conclusion of Boman’s microlocal regularity theorem [Bo 1992] is that $f$ is zero in a neighborhood of $\hat{y}_0$. (His theorem is valid if $\hat{y}_0$ is not a hypersurface, as noted on [Bo 1992, p. 1234].)

Now that we know $f$ is zero near $\hat{y}_0$, we use Lemma 3.2 to eat away further at $\text{supp } f$. If the codimension of each $\hat{y}$ in $X$ is one, then one can use the $\hat{y}$ as hypersurfaces to eat away at $\text{supp } f$ using Proposition 3.1 and Lemma 3.2 (see e.g., [BQ 1987]). However, if the codimension is greater than one, then we must construct codimension one surfaces $S$ that are “near” to $\hat{y}$ and use these surfaces to eat away at $\text{supp } f$. We construct these surfaces $S$ as small $\delta$ tubes about each $\hat{y}$.

Let $\epsilon > 0$. Let $x \in X$ and define $B_\epsilon(x)$ to be the open geodesic ball centered at $x$ of radius $\epsilon$. Now, let $T \subset X$. We define the $\epsilon$ neighborhood of $T$ to be $T_\epsilon = \bigcup_{x \in T} B_\epsilon(x)$. Choose $\epsilon_0 > 0$ so small that the closure, $K$, of the $\epsilon_0$ neighborhood of $\text{supp } f$ is compact.

Let $y_0 \in A$ be as in the statement of Theorem 2.2 and assume the conclusion of the theorem is false. Let $y_1 \in A$ be such that $\hat{y}_1 \cap \text{supp } f \neq \emptyset$, and let $s : [0,1] \to A$ be a continuous path from $y_0$ to $y_1$. By reparameterizing a part of this path, we can assume that $s(t)$ meets $K$ for all $t$ and that $\hat{y}_1$ is the only manifold in the path that meets $\text{supp } f$.

We begin to construct the hypersurfaces used to eat away at $\text{supp } f$. First, we must show that a sufficient number of covectors near each $N^*\hat{s}(t)$ are conormal to manifolds in $A$.

**Lemma 3.3.** There is an open conic set $V \subset T^*X \setminus 0$ such that, for each $t \in [0,1],$

i. $V$ is a neighborhood of each $N^*\hat{s}(t) \setminus 0$, and

ii. if $(x, \xi) \in V$, then there is a $y \in A$ with $(x, \xi) \in N^*\hat{y}$.

**Proof.** A calculation using the fact $d\pi_Y$ has maximum rank (by the Bolker Assumption) and Proposition 4.1.4 [Hö 1971, p. 167] show that the differential of $\pi_X : \Gamma \to T^*X$ is surjective. Therefore, $\pi_X$ is an open map. As $A$ is open, $\pi_Y^{-1}(A)$ is open in $\Gamma$. Therefore, $V = \pi_X\left(\pi_Y^{-1}(A)\right)$ is open. As $A$ is a neighborhood of each $s(t)$, one can see that $V$ is the desired neighborhood of each $N^*\hat{s}(t) \setminus 0$ by using the diagram (2.1) and [Q 1993a, Proposition 3.2].

For $\epsilon \in (0,\epsilon_0)$ define $S_\epsilon(t) = \text{Cl}\left(\left(K \cap \hat{s}(t)\right)_{\epsilon}\right)$.

Now choose $\epsilon_1 \in (0,\epsilon_0)$ so that:

(3.1) If $\delta \in (0, \epsilon_1)$, then $S_\delta(0)$ is disjoint from $\text{supp } f$.

(3.2) For each $x \in K$, $B_{2\epsilon_1}(x)$ is contained in a convex geodesic ball on which the geodesic distance is smooth, and this convex ball is contained normal disks centered at each of its points.

Requirement (3.1) is possible because $\hat{y}_0$ is disjoint from the compact set $\text{supp } f$; and (3.2) can be done because $K$ is compact and by [KN, Theorem 3.6, p. 166].

The following lemma provides the surfaces to eat away at $\text{supp } f$.

**Lemma 3.4.** There is a $\delta_1 \in (0, \epsilon_1)$ such that for each $t \in [0,1]$, if $x \in \text{supp } f \cap \partial S_{\delta_1}(t)$, then $\partial S_{\delta_1}(t)$ is a $C^2$ hypersurface near $x$ and, if $\xi \in N^*_xS_{\delta_1}(t) \setminus 0$, then $(x, \xi) \in V$ where $V$ is the neighborhood given in Lemma 3.3.
Proof. Let \( \delta \in (0, \varepsilon_1) \). Assume \( \partial S_\delta(t) \) meets \( \text{supp} \ f \) at a point \( x \). Then, \( x \) is \( \delta \) units from a point \( z \in \hat{s}(t) \) and both \( x \) and \( z \) are in \( K \). By (3.2), \( B_{2\varepsilon_1}(z) \) is a normal geodesic neighborhood of \( z \), and \( x \in B = B_{\varepsilon}(z) \). Using geodesic coordinates about \( z \) and (3.2), one can see that the surface \( \partial S_\delta(t) \) is the envelope of \( \delta \) balls centered at points on \( \hat{s}(t) \) near \( z \). One can calculate the envelope of the \( \delta \)-balls that make up \( S(t) \) for centers in \( B \cap \hat{s}(t) \) and show that, if \( \delta \) is sufficiently small, this envelope is smooth and all conormals to \( \partial S_\delta(t) \) are in \( V \) for points \( \delta \) units from \( B \cap \hat{s}(t) \) (start with the calculations in footnote 2 of [Q 1993b]). By choosing \( \delta \) small enough, one can ensure that \( \delta \)-balls with centers away from \( B \cap \hat{s}(t) \) do not meet this envelope. By compactness, this construction can be done for the same \( \delta \) for all points in \( \hat{s}(t) \cap K \). And, furthermore, by the choice of \( \varepsilon_1, (3.2) \), and continuity of the path \( s(t) \), and compactness of \([0, 1] \), the same \( \delta \) can be chosen for all \( t \). We let \( \delta_1 \) be such a number. [1]

By (3.1), \( S_{\delta_1}(0) \) is disjoint from \( \text{supp} \ f \) and by assumption, \( S_{\delta_1}(1) \) meets \( \text{supp} \ f \). Let \( t_1 \) be the smallest value of \( t \in [0, 1] \) such that \( S_{\delta_1}(t) \) meets \( \text{supp} \ f \). By the choice of \( t_1 \), \( S_1 = S_{\delta_1}(t_1) \) meets \( \text{supp} \ f \) only on the boundary, \( \partial S_1 \). Let \( x \in \partial S_1 \cap \text{supp} \ f \) and \( \xi \in N_2 S_1 \setminus 0 \). Then, by Lemmas 3.4 and 3.3 there is a \( y_1 \subset A \) with \( (x, \xi) \in N^* y_1 \). Now, Proposition 3.1 shows that \((x, \xi) \notin WFA(f) \). However, Lemma 3.2 gives a contradiction that proves the theorem. [1]

Proof of Proposition 2.3. We use Proposition 3.1 to show \( f \) is real-analytic in \( U \).

First, we show that wavefront set inside \( U \) is detected by the given data.

Lemma 3.5. The map \( \pi_X : \Gamma \to T^* X \setminus 0 \) is surjective.

Proof. As \( \pi_X : Z \to X \) is a fibration, \( \pi_X : \Gamma \to T^* X \setminus 0 \) is a surjection on the base. So, let \( x_1 \in X \) and let \( \Gamma_{x_1} = \{(x, \xi, y, \eta) \in \Gamma | x = x_1 \} \). Then, \( \pi_X : \Gamma_{x_1} \to T^*_x X \) is an open map by the proof of Lemma 3.3. As \( \pi_X \) is open and linear on the fibers, the restriction to sphere bundles \( P : ST^*_x X \to ST^*_x X \) is also open. (To define \( P \), view the sphere bundles as bundles of open rays from origin, and define \( P \) to take each ray in the fibers of \( \Gamma_{x_1} \) to the image ray under \( \pi_X \) in \( T^*_x X \).) As \( \pi_X : Z \to X \) is proper, \( ST^*_x X \) is compact. Now, because \( ST^*_x X \) is a sphere, \( P \) is surjective. This implies \( \pi_X \) is surjective. [1]

By Lemma 3.5, for each \((x, \xi) \in T^* U \setminus 0 \), there is a \( y \in Y \) with \((x, \xi) \in N^* y \).

By the hypotheses of the theorem and because \( y \) meets \( U \) at \( x \), \( R_\mu f \) is zero near \( y \). Therefore, \((x, \xi) \notin WFA(f) \). As this is true for arbitrary \((x, \xi) \in T^* U \setminus 0 \), \( f \) must be real-analytic on \( U \). As \( f \) is zero to infinite order at \( x_0 \), \( f \) must be identically zero in \( U \).

With the appropriate definition of flatness along a manifold [Bo 1992], the proofs of Theorem 2.2 and Proposition 2.2 are valid for distributions.

References


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