RADON TRANSFORMS SATISFYING THE BOLKER ASSUMPTION

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Abstract. Using the techniques of real-analytic microlocal analysis, we prove support theorems for Radon transforms integrating with non-zero real-analytic measures. We assume the transforms satisfy the Bolker Assumption, a microlocal injectivity condition that simplifies the microlocal analysis.

1. Introduction.

Radon transforms are integral transforms. Given a real-analytic Riemannian manifold X and a collection, Y, of submanifolds of X, one can define a Radon transform R_μ that integrates functions on X over each submanifold in Y with respect to a measure μ. Further structure on X, Y–the double fibration (2.1) [GGS, He 1984]–is needed to prove injectivity and other properties of this transform. Guillemin and Sternberg (*e.g.,* [GS]) have used this structure and microlocal analysis to understand properties of Radon transforms (see also [GU]). These advances make it natural to investigate general Radon transforms with arbitrary measures.

A *support theorem* for a Radon transform gives restrictions on the support of a function from information on the support of its Radon transform. Classical techniques have been used to prove support theorems for the transforms with canonical measures on hyperplanes [He 1964, GGV] and horocycles [He 1973]. The problem is more subtle for transforms with general measures, and support theorems have been proven under either strong symmetry conditions such as rotation invariance (*e.g.,* [Q 1983]) or strong smoothness conditions on the measure such as real-analyticity (*e.g.,* [BQ 1987]). A counterexample [Bo 1993] of a non-injective Radon transform on lines with a positive C^∞ measure shows the importance of strong hypotheses on the Radon transform for injectivity to hold.

In this article, we give a general support theorem, Theorem 2.2, that is valid for all Radon transforms with real-analytic measures that satisfy the Bolker Assumption (Definition 2.1). We also give a local support theorem, Proposition 2.3. In §2, the main definitions and theorems are given. The microlocal properties of R_μ are recalled and the main theorem are proven in §3.

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2. Definitions and main results.

Let X and Y be connected real-analytic manifolds. Let Z be a connected imbedded submanifold of $X \times Y$. The double fibration (below left) [GGS] has a corresponding diagram on the cotangent spaces (below right):

$$(2.1)\qquad \begin{array}{ccc} Z & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \pi_X} & & \\ X & & \end{array} \qquad\qquad \begin{array}{ccc} \Gamma = N^*(Z)\setminus 0 & \xrightarrow{\pi_Y} & T^*(Y)\setminus 0 \\ \downarrow{\scriptstyle \pi_X} & & \\ T^*(X)\setminus 0 & & \end{array}$$

where π_X and π_Y are the natural projections in each case. In (2.1), N^*Z is the conormal bundle of Z in $T^*(X \times Y)$. If A is a submanifold of X, then we will let N^*A denote the conormal bundle of A in T^*X.

When the left diagram in (2.1) is a double fibration, the natural projections π_X and π_Y are fibrations with connected fibers and π_X is proper [GGS, He 1984]. The double fibration defines submanifolds of X, $\hat{y} = \pi_X \circ \pi_y^{-1}\{y\}$, all of which are diffeomorphic to the fiber of π_Y. Let the weight $\mu(x,y)$ be real-analytic and nowhere zero on Z. For $f \in C_c(X)$, the Radon transform of f is defined at $y \in Y$ by

$$R_\mu f(y) = \int_{x\in\hat{y}} f(x)\mu(x,y)dm_y(x) \tag{1.2}$$

where dm_y is the measure on $\hat{y}$ induced from the Riemannian structure on X. By duality, $R_\mu : \mathcal{E}'(X) \to \mathcal{E}'(Y)$ is continuous. The key definition is:

Definition 2.1. *The double fibration* (2.1) *(or the Radon transform R_μ) satisfies the* **Bolker Assumption** *if and only if $\pi_Y : \Gamma \to T^*Y\setminus 0$ is an injective immersion.*

This is named after Ethan Bolker who formulated an analogous definition for finite Radon transforms. Our main theorem is:

Theorem 2.2. *Let the Radon transform R_μ be defined by a real-analytic double fibration that satisfies the Bolker assumption. Assume the weight μ is nowhere zero and real-analytic. Let $\mathcal{A}$ be an open connected subset of Y. Assume $f \in C_c(X)$ with $R_\mu f(y) = 0$ for all $y \in \mathcal{A}$ and assume, for some $y_0 \in \mathcal{A}$, f is zero to infinite order on $\hat{y}_0$.*[1] *Then, for all $y \in \mathcal{A}$, $\hat{y}$ is disjoint from supp f.*

This theorem implies the "standard" theorem of [BQ 1987, Q 1993a,c]: *if f is zero in a neighborhood of $\hat{y}_0$ and the other hypotheses of Theorem 2.2 hold, then f is zero on all $\hat{y}$ for $y \in \mathcal{A}$.* This weaker theorem holds for $f \in \mathcal{E}'(X)$ as can be seen from the proof in §3. The key to Theorem 2.2 which is not in these previous theorems is a microlocal regularity theorem of Boman [Bo 1992]. Boman has used this theorem to prove support theorems for hyperplane transforms and $\mathcal{S}(\mathbb{R}^n)$ [Bo 1991].

Many classical Radon transforms satisfy the Bolker assumption. Among those are the transforms in Euclidean space integrating over: $k-$planes, real, complex, and quaternionic hyperplanes, and the Cayley line, [*e.g.,* GS, Q 1993c]; the point horocycle transform on non-compact rank one symmetric spaces [Gu, Q 1993a]; and the maximal totally geodesic transforms on compact two point symmetric spaces (by

[1] That is, for each $n \in \mathbb{N}$, $f(x)$ is bounded by a fixed multiple of the n^{th} power of the geodesic distance between x and $\hat{y}_0$.

the projection argument in [Q 1983, p. 520]). Support theorems have been proven for classical measures on these spaces [*e.g.,* He 1964, GGV, Gl, He 1973] and with non-standard measures [BQ 1987 (real hyperplanes), Q 1993c (other planes), Q 1993a (rank one horocycle transform), and Q 1983, Q 1987 (transforms on compact two point homogeneous spaces)]. If the Bolker Assumption does not hold for a Radon transform, then more subtle geometric arguments (*e.g.,* [BQ 1993, GQ]) are needed to prove support theorems.

An example in [HSSW] demonstrates the necessity of some growth assumption on f near some $y_0 \in \mathcal{A}$. Using local versions of our arguments, one can easily prove local support theorems in the spirit of those in [Gl]. One such theorem (proved in §3) is:

Proposition 2.3. *Let the Radon transform R_μ be defined by a real-analytic double fibration that satisfies the Bolker assumption. Assume the weight μ is nowhere zero and real-analytic. Let $x_0 \in X$ and let U be an open connected neighborhood of x_0. Let $f \in C_c(X)$ and assume $R_\mu f(y) = 0$ for all $\hat{y}$ that meet U. Assume that f is zero to infinite order at x_0. Then, f is zero in U.*

Note that the only growth restriction on f away from x_0 is that f has compact support–U does not necessarily contain supp f. If more is known about the geometry of the manifolds $\hat{y}$, then local hole theorems such as Theorem 4.3 of [Q 1993a] are valid.

3. Proofs.

Let $\mathrm{WF_A}(f)$ denote the analytic wave front set of $f \in \mathcal{D}'(X)$ [Hö 1983]. The two keys to the proofs of the main theorem are: Proposition 3.1, the microlocal properties of R_μ; and Lemma 3.2, a theorem of Hörmander, Kawai, and Kashiwara [Hö 1983] on the analytic wave front set of a distribution at the boundary of its support.

Proposition 3.1. *Let R_μ be a Radon transform with nowhere zero real-analytic weight μ that satisfies the Bolker Assumption. Let $f \in \mathcal{E}'(X)$ and let $y_1 \in Y$. Assume $R_\mu f(y) = 0$ for all y in an open neighborhood of y_1. Let $(x,\xi) \in N^*\hat{y}_1 \setminus 0$. Then $(x,\xi) \notin \mathrm{WF_A}(f)$.*

Proof of Proposition 3.1. It is well known that if a Radon transform satisfies the Bolker Assumption, then R_μ is an elliptic Fourier integral operator associated with the Lagrangian manifold Γ and $R_\mu^* R_\mu$ is an elliptic pseudodifferential operator [*e.g.,* GS, Q 1980]. The statements of Proposition 3.1 follow immediately from the calculus of analytic elliptic Fourier integral operators [SKK], [Ka]: *As $R_\mu f(y) = 0$ for y near y_1, then $(\mathrm{WF_A} R_\mu f) \cap (T^*_{y_1} Y \setminus 0) = \emptyset$. Using the calculus of elliptic Fourier integral operators one sees that $\left[\pi_X \circ \pi_Y^{-1}(T^*_{y_1} Y \setminus 0)\right] \cap \mathrm{WF_A} f = \emptyset$. But $\pi_X\left(\pi_Y^{-1}(T^*_{y_1} Y \setminus 0)\right) = N^*\hat{y}_1 \setminus 0$ (see e.g.,* [Q 1993a, Proposition 3.2]). ▌

The next lemma allows us to eat away at supp f using hypersurfaces. It is a special case of Theorem 8.5.6 [Hö 1983].

Lemma 3.2 [Hö 1983]. *Let $S \subset X$ be a C^2 hypersurface. Let $x \in S$. Let $f \in \mathcal{E}'(X)$ and assume f is zero on one side of S locally near x (i.e., there is a neighborhood U of x such that $U \setminus S$ consists of two connected open sets, x is in the boundary of each, and f is zero on one of these sets). If $x \in$ supp f and $(x,\xi) \in N^*(\partial S) \setminus 0$, then $(x,\xi) \in \mathrm{WF_A}(f)$.*

Proof of Theorem 2.2. By the assumptions of the theorem and the conclusion of Proposition 3.1, we see that $N^*\hat{y}_0 \cap \mathrm{WF_A}(f) = \emptyset$. Therefore, as a distribution,

f can be restricted to $\hat{y}_0$. And, because f is zero to infinite order along $\hat{y}_0$, f is flat along $\hat{y}_0$ as a distribution (the proof uses the definition in [Bo 1992] and Proposition 2.5.11 of [Hö 1971] to show all distribution derivatives of f are zero along $\hat{y}_0$). Under these hypotheses ($N^*\hat{y}_0 \cap \mathrm{WF_A}(f) = \emptyset$ and f is flat along $\hat{y}_0$), the conclusion of Boman's microlocal regularity theorem [Bo 1992] is that f is zero in a neighborhood of $\hat{y}_0$. (His theorem is valid if $\hat{y}_0$ is not a hypersurface, as noted on [Bo 1992, p. 1234].)

Now that we know f is zero near $\hat{y}_0$, we use Lemma 3.2 to eat away further at supp f. If the codimension of each $\hat{y}$ in X is one, then one can use the $\hat{y}$ as hypersurfaces to eat away at supp f using Proposition 3.1 and Lemma 3.2 (see *e.g.,* [BQ 1987]). However, if the codimension is greater than one, then we must construct codimension one surfaces S that are "near" to $\hat{y}$ and use these surfaces to eat away at supp f. We construct these surfaces S as small δ tubes about each $\hat{y}$.

Let $\epsilon > 0$. Let $x \in X$ and define $B_\epsilon(x)$ to be the open geodesic ball centered at x of radius ϵ. Now, let $T \subset X$. We define the ϵ neighborhood of T to be $T_\epsilon = \cup_{x \in T} B_\epsilon(x)$.

Choose $\epsilon_0 > 0$ so small that the closure, K, of the ϵ_0 neighborhood of supp f is compact.

Let $y_o \in \mathcal{A}$ be as in the statement of Theorem 2.2 and assume the conclusion of the theorem is false. Let $y_1 \in \mathcal{A}$ be such that $\hat{y}_1 \cap$ supp $f \neq \emptyset$, and let $s : [0,1] \to \mathcal{A}$ be a continuous path from y_o to y_1. By reparameterizing a part of this path, we can assume that $\hat{s}(t)$ meets K for all t and that $\hat{y}_1$ is the only manifold in the path that meets supp f.

We begin to construct the hypersurfaces used to eat away at supp f. First, we must show that a sufficient number of covectors near each $N^*\hat{s}(t)$ are conormal to manifolds in $\mathcal{A}$.

Lemma 3.3. *There is an open conic set $V \subset T^*X \setminus 0$ such that, for each $t \in [0,1]$,*

i. V is a neighborhood of each $N^*\hat{s}(t) \setminus 0$, and
ii. if $(x,\xi) \in V$, then there is a $y \in \mathcal{A}$ with $(x,\xi) \in N^*\hat{y}$.

Proof. A calculation using the fact $d\pi_Y$ has maximum rank (by the Bolker Assumption) and Proposition 4.1.4 [Hö 1971, p. 167] show that the differential of $\pi_X : \Gamma \to T^*X$ is surjective. Therefore, π_X is an open map. As $\mathcal{A}$ is open, $\pi_Y^{-1}(\mathcal{A})$ is open in Γ. Therefore, $V = \pi_X\left(\pi_Y^{-1}(\mathcal{A})\right)$ is open. As $\mathcal{A}$ is a neighborhood of each $s(t)$, one can see that V is the desired neighborhood of each $N^*\hat{s}(t) \setminus 0$ by using the diagram (2.1) and [Q 1993a, Proposition 3.2]. ▌

For $\epsilon \in (0, \epsilon_0)$ define $S_\epsilon(t) = \mathrm{Cl}\left[\left(K \cap \hat{s}(t)\right)_\epsilon\right]$.

Now choose $\epsilon_1 \in (0, \epsilon_0)$ so that:

(3.1) If $\delta \in (0, \epsilon_1)$, then $S_\delta(0)$ is disjoint from supp f.
(3.2) For each $x \in K$, $B_{2\epsilon_1}(x)$ is contained in a convex geodesic ball on which the geodesic distance is smooth, and this convex ball is contained normal disks centered at each of its points.

Requirement (3.1) is possible because $\hat{y}_o$ is disjoint from the compact set supp f; and (3.2) can be done because K is compact and by [KN, Theorem 3.6, p. 166]. The following lemma provides the surfaces to eat away at supp f.

Lemma 3.4. *There is a $\delta_1 \in (0, \epsilon_1)$ such that for each $t \in [0,1]$, if $x \in$ supp $f \cap \partial S_{\delta_1}(t)$, then $\partial S_{\delta_1}(t)$ is a C^2 hypersurface near x and, if $\xi \in N_x^* S_{\delta_1}(t) \setminus 0$, then $(x,\xi) \in V$ where V is the neighborhood given in Lemma 3.3.*

Proof. Let $\delta \in (0, \epsilon_1)$. Assume $\partial S_\delta(t)$ meets supp f at a point x. Then, x is δ units from a point $z \in \hat{s}(t)$ and both x and z are in K. By (3.2), $B_{2\epsilon_1}(z)$ is a normal geodesic neighborhood of z, and $x \in B = B_\epsilon(z)$. Using geodesic coordinates about z and (3.2), one can see that the surface $\partial S_\delta(t)$ is the envelope of δ balls centered at points on $\hat{s}(t)$ near z. One can calculate the envelope of the δ-balls that make up $S(t)$ for centers in $B \cap \hat{s}(t)$ and show that, if δ is sufficiently small, this envelope is smooth and all conormals to $\partial S_\delta(t)$ are in V for points δ units from $B \cap \hat{s}(t)$ (start with the calculations in footnote 2 of [Q 1993b]). By choosing δ small enough, one can ensure that δ-balls with centers away from $B \cap \hat{s}(t)$ do not meet this envelope. By compactness, this construction can be done for the same δ for all points in $\hat{s}(t) \cap K$. And, furthermore, by the choice of ϵ_1, (3.2), and continuity of the path $s(t)$, and compactness of $[0,1]$, the same δ can be chosen for all t. We let δ_1 be such a number. ▌

By (3.1), $S_{\delta_1}(0)$ is disjoint from supp f and by assumption, $S_{\delta_1}(1)$ meets supp f. Let t_1 be the smallest value of $t \in [0,1]$ such that $S_{\delta_1}(t)$ meets supp f. By the choice of t_1, $S_1 = S_{\delta_1}(t_1)$ meets supp f only on the boundary, ∂S_1. Let $x \in \partial S_1 \cap$ supp f and $\xi \in N^*_x S_1 \setminus 0$. Then, by Lemmas 3.4 and 3.3 there is a $y_1 \subset \mathcal{A}$ with $(x,\xi) \in N^*\hat{y}_1$. Now, Proposition 3.1 shows that $(x,\xi) \notin \mathrm{WF_A}(f)$. However, Lemma 3.2 gives a contradiction that proves the theorem. ▌

Proof of Proposition 2.3. We use Proposition 3.1 to show f is real-analytic in U. First, we show that wavefront set inside U is detected by the given data.

Lemma 3.5. *The map $\pi_X : \Gamma \to T^*X \setminus 0$ is surjective.*

Proof. As $\pi_X : Z \to X$ is a fibration, $\pi_X : \Gamma \to T^*X \setminus 0$ is a surjection on the base. So, let $x_1 \in X$ and let $\Gamma_{x_1} = \{(x,\xi,y,\eta) \in \Gamma \big| x = x_1\}$. Then, $\pi_X : \Gamma_{x_1} \to T^*_{x_1}X$ is an open map by the proof of Lemma 3.3. As π_X is open and linear on the fibers, the restriction to sphere bundles $P : S\Gamma_{x_1} \to ST^*_{x_1}X$ is also open. (To define P, view the sphere bundles as bundles of open rays from origin, and define P to take each ray in the fibers of $\Gamma_{x_1}$to the image ray under π_X in $T^*_{x_1}X$.) As $\pi_X : Z \to X$ is proper, $S\Gamma_{x_1}$ is compact. Now, because $ST^*_{x_1}X$ is a sphere, P is surjective. This implies π_X is surjective. ▌

By Lemma 3.5, for each $(x,\xi) \in T^*U \setminus 0$, there is a $y \in Y$ with $(x,\xi) \in N^*\hat{y}$. By the hypotheses of the theorem and because $\hat{y}$ meets U at x, $R_\mu f$ is zero near y. Therefore, $(x,\xi) \notin \mathrm{WF_A}(f)$. As this is true for arbitrary $(x,\xi) \in T^*U \setminus 0$, f must be real-analytic on U. As f is zero to infinite order at x_0, f must be identically zero in U.

With the appropriate definition of flatness along a manifold [Bo 1992], the proofs of Theorem 2.2 and Proposition 2.2 are valid for distributions.

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