

CHAPTER XI

Some problems of integral geometry arising in tomography

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XI.1. Introduction

Beautiful mathematical theorems in integral geometry form the foundation of tomography. The goal of tomography is to recover the interior structure of a nontransparent body using external measurements. In this appendix, we will present an outline of a few tomographic problems that draw on the integral geometry in this book including support theorems, inversion formulas, range theorems, and microlocal analysis. We will try to show how these theorems are used in tomography. We do not intend to present here any concise survey of tomography, rather a collection of examples chosen according to their connection to the topics considered in the book and to the authors' interest. The bibliography we provide is also far from being complete, and we refer the reader to books, surveys, and collections of articles [14, 48, 49, 50, 51, 52, 67, 72, 73, 80, 124, 126, 138, 139, 150, 161, 170] for details and further references on tomography and related problems of integral geometry.

XI.2. X-ray Tomography

X-ray computed tomography (CT) is the most basic type of tomography; the model is the classical Radon transform that integrates functions over lines. However, the applied mathematics is deep and the properties of the transform provide insight into all parts of integral geometry. In this section we will discuss some of the basic ideas in X-ray CT and then talk about various types of limited data tomography. Our goal is to show some of the ways integral geometry and microlocal analysis can help one understand these problems.

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XI.2.1. Complete Data. The goal of X-ray computed tomography (CT) is to get a picture of the internal structure of an object by X-raying the object from many different directions. Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ be the density function of the object; that is, $f(x)$ is the density of the object at the point $x \in \mathbf{R}^3$. Mathematically, the goal of X-ray CT is to recover the density f from these measurements. As X-rays travel on a line L from the X-ray source through the object to an X-ray detector, they are attenuated by the material on the line they travel along (scatter can be neglected in most X-ray CT problems). If the X-rays are monochromatic, then the attenuation at $x \in L$ is proportional to the density $f(x)$ (with proportionality constant $c > 0$). If $I(x)$ is the number of X-ray photons in the beam when it arrives at x , then the intensity in a small segment of length Δx is decreased by a multiplicative factor of $cf(x)\Delta x$, so:

$$(11.1) \quad \Delta I \approx -(cf(x)\Delta x)I(x).$$

By integrating (11.1), we get:

$$\ln \left[\frac{I(\text{detector})}{I(\text{source})} \right] = c \int_L f(x) dx_L = cRf(L).$$

This integral transform R is exactly the classical Radon transform of f on the line L [71, 124]. Often, X-rays are taken in parallel slices through the object, and the problem becomes a planar problem. We will primarily consider this planar case.

Complete tomographic data are X-ray data over all lines. In practice, this means data are collected on a fairly evenly distributed set of lines throughout the object (the concept of complete data can be made precise using sampling theory as in [124, 126]). In this case, the commonly used reconstruction algorithm is filtered backprojection, which we will now discuss. We start with some standard notation. Let $\omega = (\cos \theta, \sin \theta)$ be a unit vector on the plane and p be a real number. We denote by $L(p, \omega) = \{x \in \mathbf{R}^2 \mid x \cdot \omega = p\}$ the line perpendicular to ω and p directed units from the origin. We will also often use $\omega^\perp = (-\sin \theta, \cos \theta)$, the unit vector $\pi/2$ units counterclockwise from ω . We should mention that there are two standard ways of parameterizing lines, the one we use in this section ('hyperplane' notation) and the one used in Section XI.3 ('parallel beam' notation). The theorems and ideas in each section are easier to express using the corresponding notation.

The Radon transform (11.2) of a function f in the plane can be naturally interpreted as a function of (p, ω) :

$$(11.2) \quad g = Rf(p, \omega) = \int_{x \in L(p, \omega)} f(x) dx_L.$$

We define now the backprojection operator, the dual Radon transform, as

$$R^*g(x) = \int_{S^1} g(x \cdot \omega, \omega) d\omega.$$

This is the integral of g over all lines through x , since $L(x \cdot \omega, \omega)$ is the line through x and perpendicular to ω . Let $g \in C_0^\infty(\mathbf{R} \times S^1)$, then the Fourier transform of g in the p variable

is:

$$\mathcal{F}_{p \rightarrow \sigma} g(\sigma, \omega) = \frac{1}{2\pi} \int_{p=-\infty}^{\infty} e^{-ip\sigma} g(p, \omega) dp, \quad \mathcal{F}_{\sigma \rightarrow p}^{-1} g(p, \omega) = \int_{\sigma=-\infty}^{\infty} e^{ip\sigma} g(\sigma, \omega) d\sigma$$

For $f \in C_0^\infty(\mathbf{R}^2)$ we define the two-dimensional Fourier transform by

$$\tilde{f}(\xi) = \mathcal{F}_{x \rightarrow \xi} f(\xi) = \frac{1}{(2\pi)^2} \int_{x \in \mathbf{R}^2} e^{-ix \cdot \xi} f(x) dx.$$

This allows us to define the Riesz potential I^{-1} as the operator with Fourier multiplier $|\sigma|$:

$$I^{-1}g(p, \omega) = \mathcal{F}_{\sigma \rightarrow p}^{-1}(|\sigma| \mathcal{F}_{p \rightarrow \sigma} g).$$

THEOREM 11.1. [151, 160, 124] *Let $f \in C_0^\infty(\mathbf{R}^2)$. Then $f = \frac{1}{4\pi} R^*(I^{-1}Rf)(x)$.*

The theorem is applied in practice by truncating and smoothing the multiplier $|\sigma|$ in I^{-1} and writing this truncated multiplier as a convolution operator in p [151, 124, 126]. The resulting approximate inversion algorithm becomes $f \approx R^*(\Phi *_p Rf)$ where Φ is the inverse Fourier transform of the truncated multiplier and $*_p$ denotes convolution in the p variable.

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PROOF. The proof uses some of the key elementary formulas for the Radon transform. It begins with the *Projection Slice Theorem* (which is discussed in Ch. II below (2.3)):

$$(11.3) \quad \frac{1}{2\pi} \mathcal{F}_{p \rightarrow \sigma} Rf(\sigma, \omega) = \tilde{f}(\sigma\omega).$$

Equation (11.3) is proven using Fubini's theorem. It is an elementary application of Radon transforms on spreads: to prove (11.3), integrate Rf over the spread of lines perpendicular to ω with respect to $e^{-ip\sigma} dp$ and then set $p = x \cdot \omega$ for $x \in L(p, \omega)$.

To finish the proof, one writes the two-dimensional Fourier inversion formula in polar coordinates and uses (11.3) to get:

$$\begin{aligned} f(x) &= \frac{1}{4\pi} \int_{\omega \in S^1} \int_{\sigma \in \mathbf{R}} e^{i\sigma\omega \cdot x} |\sigma| \mathcal{F}_{p \rightarrow \sigma} Rf(\sigma, \omega) d\sigma d\omega \\ &= \frac{1}{4\pi} \int_{\omega \in S^1} I^{-1}Rf(\omega \cdot x, \omega) d\omega = \frac{1}{4\pi} R^* I^{-1}Rf(x). \end{aligned}$$

□

In section XI.2.2 we will use the properties of R as an elliptic Fourier integral operator (FIO) to develop subtle results about singularity detection in tomography. Now, we will use the Projection Slice Theorem (11.3) to show R is, in fact, an elliptic FIO. Let $J : C^\infty(\mathbf{R}^2) \rightarrow C^\infty(\mathbf{R} \times S^1)$ be defined by $Jf(p, \omega) = f(p\omega)$. We take the one-dimensional inverse Fourier transform in (11.3):

$$(11.4) \quad Rf(p, \omega) = \frac{1}{2\pi} (\mathcal{F}_{\sigma \rightarrow p}^{-1} \circ J \circ \mathcal{F}_{x \rightarrow \xi})(f)(p, \omega) = \frac{1}{2\pi} \int_{\sigma \in \mathbf{R}} \int_{x \in \mathbf{R}^2} e^{i(p-\omega \cdot x)\sigma} f(x) dx d\sigma.$$

Of course, this integral does not converge absolutely, although the operation $\mathcal{F}_{\sigma \rightarrow p}^{-1} \circ J \circ \mathcal{F}_{x \rightarrow \xi}$ is defined for $f \in C_0^\infty(\mathbf{R}^2)$. To make the integrals converge, one does integrations by parts as with Fourier integral operators in general to show (11.4) can be defined on distributions [76, 168]. Equation (11.4) shows that R is an elliptic Fourier integral operator with phase function $\phi(x, p, \omega, \sigma) = (p - \omega \cdot x)\sigma$ and amplitude $1/2\pi$.

A similar exercise using the fact that $R^*Rf(x) = \left(\frac{2}{|x|}\right) * f$ shows that

$$(11.5) \quad R^*Rf(x) = \int_{\xi \in \mathbf{R}^2} e^{ix \cdot \xi} \frac{1}{\pi|\xi|} f(\xi) d\xi.$$

This shows R^*R is a classical elliptic pseudodifferential operator with symbol $1/(\pi|\xi|)$. The interplay between microlocal analysis and integral geometry is rich, and its history, beginning with Guillemin's seminal work, will be discussed in detail after Theorem 11.3.

XI.2.2. Wavefront Sets and Singularity Detection. Wavefront sets provide a subtle and elegant classification of singularities of functions and distributions, and microlocal analysis allows us to understand how Fourier integral operators transform wavefront sets.

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In particular, we will use this methodology to learn about how the Fourier integral operator R detects singularities and to predict how well reconstruction algorithms will perform when they have limited tomographic data. The wavefront set of $f \in \mathcal{D}'(\mathbf{R}^n)$ is defined in Chapter V. It is useful because it classifies singularities of f not only using points x_0 at which f is not smooth, but also *directions* above each point in which f is not smooth. We also need the concept of Sobolev wavefront set. The distribution f is in $H^s(\mathbf{R}^2)$ if and only if its Fourier transform, $\tilde{f} = \mathcal{F}_{x \rightarrow \xi} f$ is in $L^2(\mathbf{R}^2, (1 + |\xi|^2)^s)$. This relates global smoothness of f to integrability of its Fourier transform. A local version of this at a point $x_0 \in \mathbf{R}^n$ is obtained by multiplying f by a smooth cut off function $\psi \in C_0^\infty(\mathbf{R}^n)$ (with $\psi(x_0) \neq 0$) and seeing if (ψf) is in this weighted L^2 space. However, this localized Fourier transform (ψf) gives even more specific information—microlocal information—namely, the *directions* near which (ψf) is in $L^2(\mathbf{R}^2, (1 + |\xi|^2)^s)$. The precise definition is (see [134], p. 259):

DEFINITION 11.2. *A distribution f is in the Sobolev space H^s locally near $x_0 \in \mathbf{R}^n$ if and only if there is a cut off function $\psi \in \mathcal{D}(\mathbf{R}^n)$ with $\psi(x_0) \neq 0$ such that the Fourier transform $(\psi f)(\xi) \in L^2(\mathbf{R}^n, (1 + |\xi|^2)^s)$. Let $\xi_0 \in \mathbf{R}^n \setminus 0$. The distribution f is in H^s microlocally near (x_0, ξ_0) if and only if there is a cut off function $\psi \in \mathcal{D}(\mathbf{R}^n)$ with $\psi(x_0) \neq 0$ and function $u(\xi)$ homogeneous of degree zero and smooth on $\mathbf{R}^n \setminus 0$ and with $u(\xi_0) \neq 0$ such that the product $u(\xi)(\psi f)(\xi) \in L^2(\mathbf{R}^n, (1 + |\xi|^2)^s)$. The H^s wavefront set of f , $\text{WF}^s(f)$, is the complement of the set of (x_0, ξ_0) near which f is microlocally in H^s .*

It follows from this definition that, if $(x_0, \xi_0) \notin \text{WF}(f)$, then for all s , f is H^s near (x_0, ξ_0) . The wavefront set and microlocal Sobolev smoothness are usually defined on $T^*(\mathbf{R}^n) \setminus 0$, the cotangent space of \mathbf{R}^n with its zero section removed because such a definition can be extended invariantly to manifolds using local coordinates. To this end, let $x_0 \in \mathbf{R}^2$. If

$y = (y_1, y_2) \in \mathbf{R}^2$, then we let $y\mathbf{d}\mathbf{x} = y_1\mathbf{d}\mathbf{x}_1 + y_2\mathbf{d}\mathbf{x}_2$ be the cotangent vector corresponding to y in $T_{x_0}^*\mathbf{R}^2$. Now let $(p, \omega) \in \mathbf{R} \times S^1$. We let $\mathbf{d}\mathbf{p}$ and $\mathbf{d}\omega$ be the standard basis of $T_{(p, \omega)}^*(\mathbf{R} \times S^1)$. Here, $\omega = (\cos \theta, \sin \theta)$ provides coordinates on S^1 and the basic covector $\mathbf{d}\omega$ is the dual covector to $\partial/\partial\theta$. The wavefront set is extended to distributions on $\mathbf{R} \times S^1$ using these local coordinates, and it is a subset of $T^*(\mathbf{R} \times S^1)$.

The fundamental theorem that gives the relation between Sobolev wavefront of a function and its Radon transform is the following.

THEOREM 11.3. [146] *Let $f \in \mathcal{E}'(\mathbf{R}^2)$, $x_0 \in L(p_0, \omega_0)$, $\eta_0 = \mathbf{d}\mathbf{p} - (x_0 \cdot \omega_0^\perp)\mathbf{d}\omega$, and $a \in \mathbf{R}^n \setminus \mathbf{0}$. The correspondence between $\text{WF}(f)$ and $\text{WF}(Rf)$ is:*

$$(11.6) \quad (x_0; a\omega_0\mathbf{d}\mathbf{x}) \in \text{WF}(f) \quad \text{if and only if} \quad (p_0, \omega_0; a\eta_0) \in \text{WF}(Rf).$$

Given $(p_0, \omega_0; a\eta_0)$, $(x_0; a\omega_0\mathbf{d}\mathbf{x})$ is uniquely determined by (11.6). Moreover, $f \in H^s$ microlocally near $(x_0; a\omega_0\mathbf{d}\mathbf{x})$ if and only if $Rf \in H^{s+1/2}$ microlocally near $(p_0, \omega_0; a\eta_0)$. Singularities of Rf above (p_0, ω_0) give no information about singularities of f above points not on $L(p_0, \omega_0)$ or at points on this line in directions not conormal to the line.

This theorem follows from fundamental results in microlocal analysis. The general results will first be described and then the relation to Theorem 11.3 will be given and finally the proof will be outlined.

Guillemin first developed the microlocal analysis of the Radon transform. For general Radon transforms defined by double fibrations on manifolds with smooth nowhere zero measures, he proved that R is an elliptic Fourier integral operator. He also proved that R^*R is an elliptic pseudodifferential operator under the Bolker Assumption on the Lagrangian manifold associated to the Fourier integral operator R [60]. At that time, he developed a new description of Fourier integral operators using push forward and pull backs generalizing those used to define Radon transforms using double fibrations [59]. This work was included in [61]

Quinto [140] described the symbol of R^*R for all generalized Radon transforms satisfying the Bolker Assumption in terms of the measures involved into the transform. He also proved more results for the hyperplane transform, including a simple expression for the symbol and a proof of invertibility for translation invariant transforms. Beylkin [19] considered Radon transforms satisfying the Bolker Assumption integrating over surfaces in \mathbf{R}^n and analyzed the asymptotics of the symbol of $R^\dagger KR$ where R^\dagger is a backprojection related to R^* and K is a type of filter (see also [147]). We will discuss the generalized Radon transform further in Section XI.2.3.3 and Section XI.3.

The microlocal correspondence between $\text{WF}(f)$ and $\text{WF}(Rf)$ in Theorem 11.3 follows from Guillemin's seminal work [60, 61]. Because R is a FIO corresponding to a specific Lagrangian manifold, (11.6) holds. Sobolev continuity is a basic property for Fourier integral operators; any FIO, such as R , of order $1/2$ will map functions in H^s of fixed compact support continuously to functions in $H_{\text{loc}}^{s+1/2}$. If the operator is elliptic, then the function must be $1/2$ order less smooth than its image. Theorem 11.3 is a refinement of this observation. Not only does R smooth of order $1/2$ but it maps functions that are in H^s microlocally near a

given covector to functions that are in $H^{s+1/2}$ near the covector given by the correspondence (11.6).

The proof outline is as follows. For this classical Radon transform, the key to the correspondence (11.6) is (11.4), that R is an elliptic Fourier integral operator with phase function $\phi(x, p, \omega, \sigma) = (p - x \cdot \omega)\sigma$ and the explicit Lagrangian manifold associated to this operator [60, 61] (see [140] for details). The fact that $(p_0, \omega_0; a\eta_0)$ in (11.6) determines $(x_0, a\omega_0 \mathbf{d}\mathbf{x})$ is a simple exercise. The assertion about H^s is given in [146], and as discussed above, is proven using general Sobolev continuity results for FIO [76, 168].

Theorem 11.3 allows one to understand what R does to singularities in a precise and rigorous way and it provides an application of microlocal analysis to tomography. It gives an exact correspondence between singularities of f and those of Rf . Moreover, it states that the singularities of Rf that are detected from the data are of Sobolev order $1/2$ smoother than the corresponding singularities of f . For typical singularities of f (jump singularities in $H^{1/2-\epsilon}$) one can realistically expect the corresponding singularities of Rf not to be masked by noise.

The theorem has the corollary:

COROLLARY 11.4. *Data Rf for (p, ω) arbitrarily near (p_0, ω_0) detect singularities of f perpendicular to the line $L(p_0, \omega_0)$ but not in other directions. If Rf is in $H^{s+1/2}$ locally near (p_0, ω_0) , then f is microlocally in H^s at all points on $L(p_0, \omega_0)$ at all directions conormal to $L(p_0, \omega_0)$. Conversely if a direction above (p_0, ω_0) is in $\text{WF}^{s+1/2}(Rf)$ then f has H^s wavefront set conormal to $L(p_0, \omega_0)$ at the point given by (11.6).*

Corollary 11.4 follows from the correspondence (11.6) for WF and WF^s and the fact that data $\text{WF}^{s+1/2}Rf$ above (p_0, ω_0) provides information only about singularities of f conormal to $L(p_0, \omega_0)$. Palamodov stated a closely related idea in [133]. The ‘‘tangent casting’’ effects of [156] are related to Corollary 11.4.

A simple illustration will give an intuitive feeling for the corollary. Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be equal to one on the unit disk and zero outside. Then, $Rf(p, \omega) = 2\sqrt{1-p^2}$ for $|p| \leq 1$ and $Rf(p, \omega) = 0$ for $|p| > 1$. The only lines where Rf is not smooth are those with $|p| = 1$ and these lines are tangent lines to the unit circle. Since $\text{WF}(f)$ is the set of vectors normal to the unit circle, singularities of Rf precisely capture singularities of f .

XI.2.3. Limited Data Tomography. *Limited tomographic data* are tomographic data given on some proper open subset of $\mathbf{R} \times S^1$. Theorem 11.3 and Corollary 11.4 provide a paradigm to decide which singularities of f are stably visible from limited tomographic data, and we will examine what this predicts for the three common types of limited data: limited angle CT (§XI.2.3.1), exterior CT, (§XI.2.3.2), and the interior problem or local CT (§XI.2.3.3).

The basic idea is as follows. If Rf is in H^s near $(p_0, \omega_0) \in \mathbf{R} \times S^1$, then, by Corollary 11.4, f must be in $H^{s+1/2}$ at all points of $L(p_0, \omega_0)$ in directions conormal to $L(p_0, \omega_0)$. By Theorem 11.3, if Rf is not in H^s microlocally in some direction at (p_0, ω_0) , then f cannot be in $H^{s+1/2}$ conormal to $L(p_0, \omega_0)$ at a specific point on this line that is given by (11.6). So,

singularities of Rf above (p_0, ω_0) determine specific singularities of f conormal to $L(p_0, \omega_0)$. However, data Rf arbitrarily close to (p_0, ω_0) do not tell anything in a stable way about singularities of f away from $L(p_0, \omega_0)$ or at points of $L(p_0, \omega_0)$ in directions not conormal to this line. By (11.6), these singularities are visible from other data.

A typical tomographic density f is often a piecewise smooth function that is smooth on open sets with well-behaved boundaries. So, singularities of f occur at the boundaries, and the singularities are in $H^{1/2-\epsilon}$ for $\epsilon > 0$. By Theorem 11.3, singularities of the Radon data Rf will be in $H^{-\epsilon}$. A limitation of this analysis is that any discrete data Rf can be considered smooth. However, standard singularities of Rf are going to have large norm in $H^{1/2}$ and so should be visible. Moreover, the paradigm of the preceding paragraph is observed in all typical CT reconstructions from limited data.

Ramm and Zaslavsky [152] have analyzed how the Radon transform itself behaves on functions that are smooth except at smooth boundary surfaces. They give precise asymptotics of Rf at lines tangent to boundaries depending on the curvature of the boundaries, and they have proposed a singularity detection method using this information on the raw data [152]. This method has been tested on simulated data [84]. A more general method has been proposed [146].

We now apply Theorem 11.3 and Corollary 11.4 as well as some classical theorems to three common types of limited data tomography problems in the plane. We will use the microlocal results to determine which singularities of f should be visible from limited tomographic data. One can also understand stability of limited data tomography using singular value decompositions [35, 116, 107, 110, 113, 114], and they reflect the principle predicted by the microlocal analysis: singular functions associated with large singular values (which are easy to reconstruct) oscillate in directions in which wavefront is easily detectable, and vice versa.

XI.2.3.1. Limited Angle Tomography. Let U be a proper open subset of S^1 such that $U = -U$. *Limited angle data* are tomographic data given on $\mathbf{R} \times U$ only, and the limited angle Radon transform is the Radon transform $R : L_0^1(\mathbf{R}^2) \rightarrow L_0^1(\mathbf{R} \times U)$ where L_0^1 is the set of L^1 functions of compact support. The goal is to recover f using data $Rf(p, \omega)$ for $(p, \omega) \in \mathbf{R} \times U$. This problem comes up in electron microscopy [33] and in X-ray airport inspections of luggage traveling on a belt. In both cases, one cannot acquire X-ray CT data all around the object but only in a limited angular range. Alternatively, in electron microscopy, one has a picture that shows many molecules with the same crystalline structure but oriented in random directions. This is like regular CT with complete data, but with no information about the directions.

Using the projection slice theorem (11.3), one can prove this limited data transform is injective for functions of compact support: knowing Rf on $\mathbf{R} \times U$ gives the Fourier transform \tilde{f} on the cone in \mathbf{R}^2 , $V = \{t\omega \mid t \in \mathbf{R}, \omega \in U\}$. Since f has compact support, \tilde{f} is real analytic and this Fourier data determines f . Tuy [169] developed a reconstruction algorithm for this problem by extending this Fourier data \tilde{f} on the cone V to \mathbf{R}^2 by making \tilde{f} equal to zero off of V , then taking the inverse Fourier transform. This method destroys compact support since

the Fourier transforms of compactly supported functions are real-analytic. His inversion is really a pseudodifferential operator that is elliptic in directions from V so standard results about wavefront sets [76, 168] show that all wavefront of f in the directions in U will be preserved.

Other appealing and successful inversion methods involve extending the Radon data from $\mathbf{R} \times U$ to $\mathbf{R} \times S^1$ using the range conditions on the Radon transform [107, 103]. The basic idea is to project the data onto the range of the Radon transform with complete data (on $\mathbf{R} \times S^1$) and then inverting this consistently completed data using filtered backprojection. Here are the details. Let B be a ball of radius $M > 0$. To implement this, one uses the singular value decomposition for $R : L^2(B) \rightarrow L^2([-M, M] \times S^1)$ in appropriate measures [30, 124]. One projects the incomplete data onto this basis of singular functions on $L^2([-M, M] \times S^1)$. The projection is unstable because it requires an extrapolation: the singular basis is not orthogonal on $[-M, M] \times U$ but on the larger set $[-M, M] \times S^1$. The condition number of the projection matrix increases exponentially as the set U becomes smaller [107, 35]. Alternatively, if one were to construct an orthonormal basis on $L^2([-M, M] \times U)$, then the singular functions would have large norm outside of U .

This instability is intrinsic to the problem, and it is reflected in the fact that some singularities of f are not stably visible from the limited angle data. By Corollary 11.4, the only singularities of f that can be detected in a stable way are those with directions in U . To see this, choose $x \in \mathbf{R}^2$ and $\omega \in U$. Any wavefront of f at $(x; \omega \mathbf{d}\mathbf{x})$ is detected by limited angle data because the line $L(x \cdot \omega, \omega)$ is in this data set. For the same reason, wavefront of f at $(x; \omega \mathbf{d}\mathbf{x})$ for $\omega \notin U$ will not be stably detected by this limited angle data.

Moreover, an argument of Finch related to this singularity analysis shows that inversion of limited angle or exterior data is ill-posed in any range of Sobolev norms [40]. The argument for limited angle CT is as follows. We construct a non-smooth function f that is smooth at all points in \mathbf{R}^2 in all directions in U . Then, by (11.6), Rf is smooth on $\mathbf{R} \times U$. So, Rf is in every local Sobolev space above $\mathbf{R} \times U$ but f is not in all Sobolev spaces on \mathbf{R}^2 since f is not smooth. Therefore, inversion of the limited angle Radon transform is not continuous in any range of Sobolev norms.

Here is one way to construct such an f . Let $\omega \notin U$, let L be the line through the origin normal to ω , and let H be a half-plane with boundary L . Take a smooth nonzero radial function of compact support, ψ , and let f be the result of setting ψ to zero on H . Then, f is smooth at all points in \mathbf{R}^2 in all directions in U but f is not in C^∞ . In fact, $\text{WF}(f)$ is a subset of points in L with direction $\omega \mathbf{d}\omega$, that is conormals to L .

This instability is also illustrated by the singular functions for the limited angle problem [107]. Those corresponding to large singular values (easy to reconstruct) oscillate generally in directions in U and those corresponding to small singular values (hard to reconstruct) oscillate generally in directions outside of U .

XI.2.3.2. The Exterior Problem. Let $M > 1$ and assume $\text{supp } f \subset \{x \in \mathbf{R}^2 \mid |x| \leq M\}$. **Leon, check chapter reference below to book: VEZI!** In the exterior problem, one has data $Rf(p, \omega)$ for all ω but only for $|p| > 1$. By the support theorem for the Radon transform ([67], see also Chapter II, (2.66) of the book), one can reconstruct $f(x)$ for $|x| > 1$. This

problem comes up in studies of the solar corona [5] in which data are total intensities of the corona of the sun along lines exterior to the core of the sun that go from the solar corona to the observer on earth. Exterior data also occur in industrial tomography of very large objects, for which X-ray data through their centers is too highly attenuated to be usable [148].

Let $|x| > 1$ and $\omega \in S^1$. Then the only singularities of f at x that are reconstructed in a stable manner are those for ω with $L(x \cdot \omega, \omega)$ in the data set, that is for $|x \cdot \omega| > 1$. Other singularities of f are not stably detected. Because some wavefront directions are not stably detectable by exterior data, inversion for the exterior problem is highly ill-posed; a translate of Finch's example in §XI.2.3.1 can be used to show inversion of the exterior transform is discontinuous in any range of Sobolev norms. Lissianoi [105] has extended Finch's Sobolev discontinuity result to show that in the exterior problem even recovery of the function in a smaller ring than where the data are given does not help to improve stability. However, logarithmic stability has been proven by Isakov-Sun [78].

Lewitt and Bates [103], Louis [108], and Natterer [120] have developed good reconstruction algorithms that use exterior data. Lewitt and Bates' algorithm completes the exterior data by projecting it on the range of the complete Radon transform as discussed in §XI.2.3.1. The projection step is unstable because the singular functions are not orthogonal on the annulus but on a disk. Natterer's algorithm is an effective regularization method. Quinto has developed an exterior reconstruction algorithm which employs a singular value decomposition [134] for the Radon transform on domain $L^2(\{x \in \mathbf{R}^2 \mid |x| \geq 1\})$ and *a priori* information about the shape of the object to be reconstructed. Reconstructions for 'medical' phantoms are in [142] and those for industrial phantoms are in [143, 144] and real industrial data in [148]. Exactly those singularities that are supposed to be stably reconstructed are clearly imaged in the reconstruction. In Quinto's reconstruction algorithm, singularities that are not 'visible' are blurred.

XI.2.3.3. The Interior Problem and Local Tomography. Let $M > 1$ and assume $\text{supp } f \subset \{x \in \mathbf{R}^2 \mid |x| \leq M\}$. *Interior tomographic data* are data $Rf(p, \omega)$ for all ω but only for $|p| < 1$. Data are missing over lines outside the unit sphere, even though $\text{supp } f$ can meet the annulus $\{x \in \mathbf{R}^2 \mid 1 \leq |x| \leq M\}$. The goal of interior CT is to reconstruct information about $f(x)$ for $|x| < 1$. This problem comes up whenever scientists want information only about some portion of the object to be reconstructed, not the whole object, or in problems, such as high-resolution tomography of very small parts of objects or electron microscopy, for which it is difficult or impossible to get complete high-resolution CT data.

Simple examples (derived using the range theorem for the Radon transform) show the interior transform, the Radon transform with this limited data, is not injective. However, according to Corollary 11.4, one can detect all singularities of f in $|x| < 1$. To see this, choose a point x inside the unit disk and choose a direction $\omega \in S^1$. Then the line through x and normal to ω is in the data set for interior tomography. Therefore, by Theorem 11.3, any singularity of f at $(x; \omega \mathbf{d}\mathbf{x})$ is stably detected by interior data.

Maass [114] has developed a singular value decomposition for this problem. See also [110]. The authors of [103] have developed a reconstruction method that projects the interior data onto the range of the Radon transform with complete data and then inverts this completed data. Because of the non-uniqueness of the problem, this projection step is not unique. However, Maass has shown that singular functions associated to small singular values are fairly constant inside the region of interest, the unit ball [114]. These singular functions corresponding to small singular values are difficult to reconstruct, but they do not add much detail inside the region since they are relatively constant there. This reflects the fact that all Sobolev singularities inside the region of interest are stably reconstructed.

Even though the interior transform is not invertible, we have shown that all singularities in the region can be stably reconstructed. This explains why singularity detection methods work so well for this problem. Lambda tomography [160, 38, 39, 171] is a well developed algorithm that finds singularities of a function using interior data over a very small region. One starts with the filtered backprojection inversion formula, $f = \frac{1}{4\pi} R^* I^{-1} R f$ and replace I^{-1} by $I^{-2} = I^{-1} \circ I^{-1} = -\frac{d^2}{dp^2}$. The beauty of this idea is that the formula

$$(11.7) \quad \sqrt{-\Delta} f = R^* I^{-2} R f = -R^* \left(\frac{d^2}{dp^2} R f \right)$$

is a local reconstruction formula. This is true because one needs only data $Rf(p, \omega)$ near $(x \cdot \omega, \omega)$ to calculate $\left(\frac{d^2}{dp^2} R f \right)(x \cdot \omega, \omega)$ and then to calculate $R^* \frac{d^2}{dp^2} R f(x)$. Moreover, $\sqrt{-\Delta}$ is an elliptic pseudodifferential operator and so $\text{WF}(f) = \text{WF}(\sqrt{-\Delta} f)$ and $\text{WF}^s(f) = \text{WF}^{s-1}(\sqrt{-\Delta} f)$. This reconstruction formula takes singularities of f and makes them more pronounced; any singularity of f in H^s becomes a singularity in H^{s-1} because Λ is an elliptic pseudodifferential operator of order 1. At the same time, there is a cupping effect at the boundaries. Because of this, the developers of Lambda CT chose to add a multiple of $R^* R f = \Lambda^{-1} f$ to the reconstruction to smooth it out; moreover, with a good multiple, the cupping effects are decreased [38, 39].

For the X-ray transform over a curve in \mathbf{R}^3 , Louis and Maass [109] have developed a very promising generalization of Lambda CT. Greenleaf and Uhlmann [56, 57] completely analyzed the microlocal properties of $R^* R$ for admissible transforms on geodesics, including this example. The microlocal analysis of this three-dimensional Lambda CT operator has been investigated by Finch [42], Kastelevich [83], and Lan [102]. The operator $R^* \Lambda R$ adds singularities to f because R is not well enough behaved; for the classical line transform in the plane, $R^* \Lambda R$ is an elliptic pseudodifferential operator and therefore preserves singularities. The theorem corresponding to Theorem 11.3 for the X-ray transform on lines through a curve is given in [146], and it explains which singularities are stably reconstructed by this transform. The prediction is observed in the reconstructions in [109]. The general analysis in [62] and [56] provides the microlocal results needed to prove this theorem. The microlocal properties of this line transform were also studied in [22] as a basis for support and uniqueness theorems for the X-ray transform.

Limited angle and exterior versions of Lambda CT have been developed, and they are promising on simulations [94] and tests on industrial and electron microscope data (E.T. Quinto unpublished).

Another way to deal with interior data is to truncate the data to zero by multiplying by a cutoff function which is equal to one in the region of interest. If one now applies a standard Radon inversion formula to the truncated data, one can check that the result is a pseudodifferential operator with the symbol equal to one in the local region of interest. This means that although the exact values of the function are not recoverable from the local data, its singularities are, by simple usage of truncated data in inversion formulae. One sees, for instance, that the values of jumps of the function are recovered correctly [86, 92]. This idea was further developed by Katsevich and Ramm [85] under the name of pseudolocal tomography. The authors of [39] have provided a different method to calculate the jumps of f at boundaries using Lambda CT.

Madych [115] used wavelet analysis to show the strong relationship between Lambda CT and pseudolocal CT and regular filtered backprojection. Authors of the papers [36, 153] have also used wavelet techniques for local tomography. They use Radon data to calculate wavelet coefficients of the density to be reconstructed. Although they need some data slightly outside the region of interest, their methods are fairly local.

Candes and Donohoe [27] have recently defined ridgelets, wavelets that are not radially symmetric and are more sensitive to singularities in specified directions. They have developed a tomographic reconstruction algorithm using ridgelets that provides high-quality reconstructions in practice. Their ideas reflect the singularity detection predictions of Theorem 11.3.

Technicians currently use the sinogram, the graph of $Rf(p, \omega)$ in rectangular coordinates, to find boundaries, but this method is subjective, and some industrial data are too homogeneous for this to work [149].

Local tomography has been developed for generalized Radon transforms [94], such as the attenuated Radon transform discussed in §XI.3 and in particular in SXI.3.1. If $\nu = \nu(x, \omega)$ is a nowhere-zero smooth weight, we define the generalized Radon transform of $f \in C_0^\infty(\mathbf{R}^2)$ to be

$$(11.8) \quad R_\nu f(p, \omega) = \int_{x \in L(p, \omega)} f(x) \nu(x, \omega) dx_L,$$

the integral of f on the line $L(p, \omega)$ in weight ν (note that the weight ν is determined by x and ω because $x \cdot \omega = p$ when $x \in L(p, \omega)$).

An important generalized Radon transform is the attenuated transform T_μ defined in Section XI.3, (11.11). The weight for this transform is

$$\nu(x, \omega) = e^{-\int_{\tau=0}^\infty \mu(x - \tau\omega^\perp) d\tau}$$

where $\mu(y)$ is the attenuation coefficient of the body at the point y .

The generalized dual transform of R_ν is

$$R_\nu^* g(x) = \int_{\omega \in S^1} g(x \cdot \omega, \omega) \nu(x, \omega) d\omega$$

for $g \in C^\infty(\mathbf{R} \times S^1)$. Let $\mu(x, \omega)$ and $\nu(x, \omega)$ be nowhere zero smooth functions on $\mathbf{R}^2 \times S^1$.

We use this to develop a generalized Lambda tomography. The calculus of Fourier integral operators shows that the generalized Lambda operator (compare with (11.7))

$$(11.9) \quad R_\nu^* \frac{d^2}{dp^2} R_\mu$$

is an elliptic pseudodifferential operator of order one with principal (top-order) symbol [94]

$$\nu(x, \xi/|\xi|) \mu(x, \xi/|\xi|) |\xi|.$$

The analogous theorem was proven about $R_\nu^* R_\nu$ in [140]. The operator defined by (11.9) is a local operator for the same reasons the Lambda operator (11.7) is. Because $R_\nu^* \frac{d^2}{dp^2} R_\mu$ is an elliptic pseudodifferential operator, it preserves wavefront set and

$$\text{WF}^s(f) = \text{WF}^{s-1}(R_\nu^* \frac{d^2}{dp^2} R_\mu f).$$

Good reconstructions of simulations using this operator from limited angle, exterior, and interior data are given in [94]. In addition, Katsevich [81] uses properties of pseudodifferential operators to provide an algorithm to detect the values of density jumps at boundaries of regions using this operator (11.9).

In general, the attenuated Radon transform will have a measure that is not smooth on boundaries of regions at which the attenuation changes. This gives rise to pseudodifferential operators $R_\nu^* \frac{d^2}{dp^2} R_\nu$ that have non-smooth symbols. Katsevich studied these operators, and he proved that one can detect jumps of f that are not masked by singularities of the measure using this information, if one knows the measure [82]. This work is important even if one does not know the measure ν exactly. Because the operator is an elliptic pseudodifferential operator (albeit with a non-smooth symbol), the singularities of f that are not masked by singularities of ν will still be visible from the data, even if the measure is unknown. One practical issue would be whether the large amount of noise in SPECT data might create problems for any singularity detection algorithm that ‘overdifferentiates’ the data.

XI.3. Attenuated and exponential Radon transforms

Some problems of medical tomography, radiation therapy, and industrial nondestructive testing naturally lead to consideration of a special type of weighted X-ray transforms. We will briefly describe one such model, referring the reader to publications [24], [31], [32], [93], [124], [138], [139], and [158] for further details, examples, and references.

The so called Single Photon Emission Computed Tomography (SPECT) deals with the situation when a nontransparent body contains a distributed radiation source. The goal is to reconstruct the interior intensity distribution $f(x)$ of the radiation sources by conducting exterior measurements of the intensity of outgoing radiation. Let us denote by $\mu(x)$ the linear attenuation coefficient (or *absorption*) of the body at the point x . This means that

due to the absorption a beam passing through the point x suffers at a small distance Δx the relative intensity loss equal to $\mu(x)\Delta x$. Then assuming that effects of scatter are small and hence can be neglected (which is in fact not always true), the stationary one-velocity transport equation can be written as

$$(11.10) \quad \nabla_x u \cdot \nu_\phi + \mu u = f.$$

Here $u(x, \phi)$ is the density of particles at x moving in the direction of the vector $\nu_\phi = (\cos \phi, \sin \phi)$, In other words,

$$u_{x_1} \cos \phi + u_{x_2} \sin \phi + \mu u = f.$$

Exterior detectors are placed on the boundary. They are collimated in order to detect only the particles following a specific oriented line L . In fact, perfect collimation is impossible, so the detector is affected by all rays from a solid angle around the line L , but we will disregard this circumstance. Assuming that there are no particles entering the region from outside, one can easily solve the transport equation for the intensities measured at the detectors. This gives as the total detected intensity of the beam L the expression

$$(11.11) \quad T_\mu f(L) = \int_L f(x) e^{-\int_{L_x} \mu(y) dy} dx.$$

Here L_x is the segment of the line L between the point x and the detector and dy denotes the standard linear measure on L .

The operator T_μ is said to be the *attenuated X-ray transform* of the function $f(x)$ and $\mu(x)$ is called the attenuation. The formula above is a non-parametric version of this transform. One can also introduce a parametric one. As mentioned in the last section, we will use a slightly different parameterization of lines in this section than we did in section XI.2. The theorems and ideas in each section are easier to express using the respective notation. Namely, let ω be the unit vector parallel to the direction of the oriented line L , ω^\perp be its orthogonal hyperplane, and $y = L \cap \omega^\perp$. Then the oriented lines L are in one-to-one correspondence with pairs (ω, y) , where $y \in \omega^\perp$. Now

$$T_\mu f(\omega, y) = \int_{-\infty}^{\infty} f(y + t\omega) e^{-\int_t^\infty \mu(y + \tau\omega) d\tau} dt.$$

One notices that in contrast with the standard X-ray transform $f \rightarrow \int_L f(x) dx$, (11.2), the resulting function depends on the orientation of the line L . In practice one often averages over the two orientations, thus arriving to a function of non-oriented lines.

In many algorithms one assumes that the attenuation is constant inside the body and zero outside. Let us also assume that the body has a known convex shape. In this case, according to [117], the attenuated transform can be reduced to a simpler one. Indeed, the integral

$$\mathcal{D}\mu(\omega, y + t\omega) = \int_t^\infty \mu(y + \tau\omega) d\tau,$$

is the so called *divergent beam transform* of μ (the integral of μ over the ray in direction ω starting at $y + t\omega$), and it can be evaluated explicitly when μ is constant:

$$\int_t^\infty \mu(y + \tau\omega) d\tau = \mu s(\omega, y) - \mu t.$$

Here $s(\omega, y)$ is the value $\tau \geq t$ for which $y + \tau\omega$ belongs to the boundary of the body. Since the shape of the body is known, $s(\omega, y)$ is a known function. Then the transform T_μ can be rewritten as follows:

$$T_\mu f(\omega, y) = e^{-\mu s(\omega, y)} \int_{-\infty}^\infty f(y + t\omega) e^{\mu t} dt = e^{-\mu s(\omega, y)} R_\mu f(\omega, y).$$

Here the function $e^{-\mu s(\omega, y)}$ is known and

$$R_\mu f(\omega, y) = \int_{-\infty}^\infty f(y + t\omega) e^{\mu t} dt$$

is the so called *exponential X-ray transform* of the function f . (In order to avoid confusion, the reader should notice that we use notations for these transforms different from the ones in [124].) It is necessary to mention that one needs to require that the function $f(x)$ decays at infinity exponentially and sufficiently fast to offset the effect of the exponential weight in the integral.

Among the natural questions to ask about the attenuated and exponential X-ray transforms are:

Injectivity: Can a function of a natural class be reconstructed from its attenuated or exponential transforms? In other words, are these operators injective?

Inversion formulae in cases when injectivity is established.

Stability of inversion.

Range: Are these operators surjective in natural functional classes? Judging by the precedent of the standard Radon transform, the reader certainly should feel that the answer is probably negative, and some nontrivial range conditions should be satisfied. Then the question arises of describing the ranges of these operators.

Simultaneous reconstruction of the sources density f and attenuation μ . In most cases not only the value of the intensity $f(x)$ distribution, but the attenuation coefficient $\mu(x)$ as well is unknown. Is it possible to extract any information about both functions from the values of $T_\mu f$ or $R_\mu f$?

These problems will be briefly addressed below. We would also like to mention papers [37, 93, 157, 131, 132], where some interesting relations between the properties of the exponential transform and complex analysis are discovered that are beyond the scope of this text.

XI.3.1. Uniqueness of reconstruction. Uniqueness of reconstruction is obviously the first question one should ask. Non-uniqueness would mean that one is unable to recover the interior structure of the object of interest from the exterior information. When we discuss uniqueness we will assume that the function f to be reconstructed has compact support.

The problem of uniqueness for the attenuated X-ray transform in dimension two has proven to be hard. The first results on uniqueness were the local ones. Since for smooth μ the operator T_μ (after averaging over opposite directions) is an elliptic FIO (Fourier Integral Operator) [60, 61, 140, 19], one concludes that for each point x there is a neighborhood U of x such that uniqueness holds for functions f with support in U . The idea is that locally the attenuated transform looks almost like the standard X-ray transform. It was shown in [118] that C^2 smoothness of the attenuation μ is sufficient. The paper [66] contains the proof of the following statement. Let us require that the supports of all functions f under consideration belong to a fixed bounded domain Ω . Assume, for instance, that Ω is the unit ball centered at the origin. Then the transformation T_μ is a semi-Fredholm operator between the Sobolev spaces $H_0^s(\Omega)$ and $H^{s+1/2}(\mathbf{S}^1 \times [-1, 1])$ for any s (see also [124]). This means that the range of this operator is closed, and its kernel $\text{Ker}T_\mu$ is finite dimensional. In the class of such operators property $\text{Ker}A = 0$ is stable under perturbations of small operator norm. This observation immediately leads to local uniqueness result: for small Ω operator T_μ is norm close to the X-ray transform, and hence has zero kernel. The next step was made by D. Finch [41], who managed to obtain a ‘semi-local’ uniqueness results. Namely, he proved uniqueness under the condition $\|\mu\|_\infty \text{diam}\Omega < 5.37$. This result is sufficient for many practical situations. For instance, in medical applications it restricts the diameter of an object to 35.8cm. The proof is nontrivial and involves energy estimates. A breakthrough came recently in brilliant works [8, 128, 129], where positive solution of the uniqueness problem under some mild smoothness condition on the attenuation was obtained ([8]) and an explicit inversion formula found ([128, 129]) (see also [26, 43, 58, 96, 98, 125] for different derivations and implementations). We will discuss the inversion formulas in a little bit more detail in the next section.

It was noticed in [41] that a similar uniqueness problem in dimensions three and higher is trivial. The reason is that the 3D attenuated X-ray transform is the collection of the corresponding 2D transforms in all affine planes. If there is a compactly supported function $f(x)$ annihilated by the transform, then we can choose a plane which just barely touches the support of f . In this plane one can use local uniqueness theorem to conclude that the function is in fact zero there. Proceeding this way further, one can “eat the support away” and finally conclude that it is empty.

The problem of uniqueness was also considered for transforms with more general positive weights ν , the generalized Radon transform (11.8). Here there are local uniqueness results for fairly arbitrary weights and global uniqueness theorems for some special types of weights (translation invariant, rotation invariant) [140, 141, 100]. In the case of an analytic weight uniqueness in the class of compactly supported functions follows from analytic ellipticity of the corresponding FIO [21]. Analytic ellipticity has been used to prove support theorems and uniqueness in many settings such as for Radon transforms on geodesic spheres in real-analytic manifolds [145] and for circular transforms related to the wave equation (e.g., [2]) (see also [1] and [172] for related results). However, as the famous counterexample by

Boman [20] shows, the condition of infinite smoothness of the weight w function alone does not guarantee uniqueness.

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Let us move now to the much simpler case of the exponential transform R_μ . One of the most important (albeit simple) properties of the X-ray transform is the Fourier-slice (also called projection-slice) theorem (see Ch. II and (11.3)). There is an analog of this property for the exponential transform [121, 124]. Namely, let f be a compactly supported function on the plane. A straightforward calculation shows that

$$(11.12) \quad \widehat{R_\mu f}(\sigma, \omega) = \sqrt{2\pi} \tilde{f}(\sigma\omega + i\mu\omega^\perp).$$

where the hat in the left hand side denotes the one-dimensional Fourier transform with respect to the linear variable s , while tilde in the right hand side denotes the two-dimensional Fourier transform. This means that the one-dimensional Fourier transform of the projection data $R_\mu f$ provides the values of the Fourier transform of the function f on a surface in \mathbf{C}^2 . One can see this as an indication of possibility to use methods of theory of functions of several complex variables. Results of [121, 117, 44, 88, 91, 93, 3, 131, 132, 37] show that the relation between two theories is indeed very deep. We will try to show this in our exposition.

Let us turn now to the problem of uniqueness of reconstruction, i.e., injectivity of the operator R_μ . We will follow here considerations of [117] and [74]. Let us assume that function f is either compactly supported or decays as $\exp(-(\mu + \varepsilon)|x|)$ for a positive ε . Consider the surface

$$S_\mu = \{z = \sigma\omega + i\mu\omega^\perp \mid \sigma \in \mathbf{R}, \omega \in \mathbf{S}^1\} \subset \mathbf{C}^2.$$

It is straightforward to check that except the points where $\sigma = 0$ this surface is a totally real smooth two-dimensional submanifold of \mathbf{C}^2 . This means, in particular, that it is a uniqueness set for analytic functions. The formula (11.12) shows that knowing $R_\mu f$ one can recover the values of the Fourier transform \tilde{f} of the function f on the surface S_μ . Due to our decay assumption, \tilde{f} is analytic in a tubular neighborhood of \mathbf{R}^2 containing S_μ . This implies that $R_\mu f$ determines \tilde{f} and hence f uniquely.

XI.3.2. Inversion formulas. As soon as injectivity is established, it is natural to look for inversion formulas. By now inversion formulas have been established for the attenuated Radon transform (see [8, 128, 129] and also [26, 43, 58, 95, 96, 98, 125] for further discussion and numerical implementation). We will address those below in this section. We would like to start with the much simpler, but still very rich example of the exponential transform.

XI.3.2.1. *Exponential transform.* The first explicit inversion formula for the exponential X-ray transform in the plane was provided by O. Tretiak and C. Metz in [167] (see also its discussion in [124]). An inversion procedure was also provided in [12]. Let us introduce the dual exponential X-ray transform (or exponential backprojection) $R_\mu^\#$ as follows: applied to

a function $g(s, \omega)$ it produces a function on the plane according to

$$(R_{\mu}^{\#}g)(x) = \int_{\mathbf{S}^1} e^{\mu x \cdot \omega^{\perp}} g(x \cdot \omega, \omega) d\omega.$$

One can verify the formula

$$R_{-\mu}^{\#}R_{\mu}f = \left(2 \frac{\cosh \mu |x|}{|x|}\right) * f.$$

Hence, in order to reconstruct the function f from $R_{-\mu}^{\#}R_{\mu}f$ one needs to perform a deconvolution. This can be done by using appropriate generalized Riesz type potentials on the line. Namely, let

$$\zeta_{\mu}(\sigma) = \begin{cases} |\sigma| & \text{when } |\sigma| > |\mu| \\ 0 & \text{otherwise} \end{cases}$$

and I_{μ}^{-1} (a generalized Riesz potential) be the Fourier multiplier by $\zeta_{\mu}(\sigma)$.

THEOREM 11.1. [167] *Let $f \in C_0^{\infty}(\mathbf{R}^2)$. Then*

$$(11.13) \quad f = \frac{1}{4\pi} R_{-\mu}^{\#} I_{\mu}^{-1} R_{\mu} f.$$

The original proof of this filtered backprojection type formula in [167] and the proof provided in [124] are interesting and instructive, although rather technical. We will now present a different simple approach based on complex analysis [91, 93].

Let us go back to the projection-slice formula (11.12). As we have already discussed, it implies that given $R_{\mu}f$ one can obtain the values of the Fourier transform of the unknown function f on the surface S_{μ} in \mathbf{C}^2 . If instead of S_{μ} we dealt with \mathbf{R}^2 , we would just use the Fourier inversion formula to find f . So, the question is whether one can develop a Fourier inversion formula that uses the data on S_{μ} instead of \mathbf{R}^2 . The simple idea is to try to use Cauchy theorem in order to deform the surface of integration from \mathbf{R}^2 to S_{μ} . Let us look at the standard Fourier inversion formula in \mathbf{R}^2 :

$$f(x) = \int_{\mathbf{R}^2} \tilde{f}(\xi) e^{ix \cdot \xi} d\xi$$

(an additional multiplicative constant may arise depending on normalization of the Fourier transform). Assume that f is compactly supported or decays sufficiently rapidly exponentially, so $\tilde{f}(\xi)$ is analytic in all areas of interest. Consider the holomorphic differential form $\Phi_x = \tilde{f}(z) e^{ix \cdot z} dz_1 \wedge dz_2$ in \mathbf{C}^2 . Then the Fourier inversion formula reads

$$f(x) = \int_{\mathbf{R}^2} \Phi_x.$$

Now it is clear that using Cauchy theorem is a good idea. There are, however, two complications. The first one is minor and can be easily dealt with. The surface S_{μ} when projected to \mathbf{R}^2 by $z \rightarrow Re(z)$ covers $\mathbf{R}^2 \setminus \{0\}$ twice (since a point $\sigma\omega$ can be also represented as $(-\sigma)(-\omega)$). To treat this problem we can restrict the values of σ in the definition of S_{μ} to

nonnegative numbers only. The second complication is much more serious and interesting. The surface S_μ is not homological to \mathbf{R}^2 , in particular since it is not simply-connected. Namely, the points where $\sigma = 0$ constitute the boundary of a circular hole in the surface. This hole is the disk of radius μ in the imaginary plane. The idea to paste this disk to the surface does not work, since the values of \tilde{f} on this disk are not known. The hope is to find a different disk that being pasted to the surface would fix this problem. However, no matter what kind of disk we use, the values of \tilde{f} on this disk will still be unknown. What difference then does the choice of the disk make? Assume that we found an analytic disk, then the holomorphic $(2, 0)$ -form Φ_x would vanish on this disk. Thus, for an analytic disk the integral of the form Φ_x over this disk vanishes independently on the values of \tilde{f} . So, we are led to a classical problem of complex analysis about pasting analytic disks to a totally real surface in \mathbf{C}^2 . A little bit of experimentation shows that such a disk Γ_1 does exist and can be described as follows:

$$(11.14) \quad \Gamma_1 = \{z = \sigma(\omega + i\omega^\perp) \mid 0 \leq \sigma \leq \mu, \omega \in \mathbf{S}^1\}.$$

It is contained in the complex line $z_2 = iz_1$ and hence is analytic. We define now the following part of the surface S_μ :

$$\Gamma_2 = \{z = \sigma\omega + i\mu\omega^\perp \mid \sigma \geq |\mu|, \omega \in \mathbf{S}^1\} \subset S_\mu.$$

Defining $\Gamma = \Gamma_1 \cup \Gamma_2$, applying the Cauchy theorem, and using the equality $\int_{\Gamma_1} \Phi_x = 0$, we get an inversion formula

$$f(x) = \int_{\mathbf{R}^2} \Phi_x = \int_{\Gamma} \Phi_x = \int_{\Gamma_1} \Phi_x + \int_{\Gamma_2} \Phi_x = \int_{\Gamma_2} \Phi_x.$$

Writing the integral $\int_{\Gamma_2} \Phi_x$ explicitly using the definition of the form and coordinates σ and ω on S_μ , one arrives to the inversion formula (11.13). The details are easily workable and can be found in [91, 93, 158].

The idea of using Cauchy theorem and Cauchy formula for inverting the exponential X-ray transform was initially used in papers [121, 117, 44], and developed in the presented form in [91, 93]. This approach leads to a wide variety of inversion formulas [93, 158], including cases when the attenuation μ depends on the direction ω [93, 158] and when imperfect detector collimation is taken into account [91]. Invertibility of the exponential X-ray transform was also studied analytically and numerically by different methods in [63, 64, 65]. One might also be interested in related inversion formulas available for the cases of half-view [127], 3D limited data acquisition [97], and imperfect collimation [91].

Another interesting relation with complex analysis is the following. Possibility of pasting disks to totally real surfaces in \mathbf{R}^2 is a popular topic of complex analysis (e.g. [55]). In particular, the question of rigidity of such a pasted disk has been studied. Rigidity means absence of parametric deformation of the pasted analytic disks. The natural way the disk (11.14) pasted to the surface S_μ appears, suggests that one might expect its rigidity. This happens to be true [112]. Moreover, a general totally real surface in \mathbf{C}^2 in a neighborhood of the boundary of a pasted analytic disk can be reduced to a canonical form resembling

definition of S_μ , which in turn leads to the possibility of counting the number of parameters in deformations of this disk [112]. This provides a different approach to the known results ([55]) concerning this problem.

XI.3.2.2. *Attenuated transform.* We follow in this section the technique introduced in [8, 25] with an additional development from [95]. Let as before $f(x)$ be the sources intensity in a planar domain Ω , $u(x, \phi)$ - density of particles at x moving in the direction of $\nu_\phi = (\cos \phi, \sin \phi)$, and $\mu(x)$ be the linear attenuation coefficient. When scattering is absent, the transport equation (11.10) can be rewritten as

$$u_{x_1} \frac{1}{2}(e^{i\phi} + e^{-i\phi}) + u_{x_2} \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) + \mu u = f.$$

Using the complex coordinate $z = x_1 - ix_2$ one gets

$$(11.15) \quad e^{i\phi} \frac{\partial u}{\partial \bar{z}} + e^{-i\phi} \frac{\partial u}{\partial z} + \mu u = f,$$

or when the attenuation is zero,

$$(11.16) \quad e^{i\phi} \frac{\partial u}{\partial \bar{z}} + e^{-i\phi} \frac{\partial u}{\partial z} = f$$

The boundary data $u(x, \phi)|_{x \in \partial\Omega}$ is known. Let us consider the spaces $L_2(S)$ and $H^2 \subset L_2(S)$ (the Hardy space) on the circle and the orthogonal projector $P : L_2(S) \rightarrow H^2$. We also denote by $u_k = u_k(x)$ the Fourier coefficients of $u(x, \phi)$. Let R be the right shift in H^2 : $(Ru)_k = u_{k-1}$ and $L = R^*$ be the left shift. Then we easily obtain from (11.16) that

$$(u_{-1})_{\bar{z}} + (u_1)_z = f$$

or

$$(11.17) \quad 2\text{Re}((u_1)_z) = f$$

and

$$(11.18) \quad (u_k)_{\bar{z}} + (u_{k+2})_z = 0, k = 0, 1, 2, \dots$$

Let $v = Pu = v_e \oplus v_o$, where $v_e = (v_0, v_2, \dots)$ and $v_o = (v_1, v_3, \dots)$. Then (11.18) can be rewritten as

$$(11.19) \quad v_{\bar{z}} + L^2 v_z = 0,$$

which splits into independent equations for v_e and v_o :

$$(v_e)_{\bar{z}} + L^2 (v_e)_z = 0, \quad (v_o)_{\bar{z}} + L^2 (v_o)_z = 0.$$

The boundary values of both vectors v_e and v_o at $\partial\Omega$ are known. According to (11.17), only the odd part v_o plays a role in the recovery of $f(x)$.

Let us introduce a generalized ∂ -bar operator:

$$D = \frac{\partial}{\partial \bar{z}} + L^2 \frac{\partial}{\partial z}.$$

The equation (11.19) is a generalized ∂ -bar equation for the function v . If one could prove uniqueness of its solution under given boundary conditions, this would immediately imply uniqueness of recovery of f , and hence uniqueness for the attenuated Radon transform. This would follow, for instance, from a Cauchy type formula for such “analytic” vector valued functions (if $A = L^2$, such functions are called A -analytic in [8, 25]). This is exactly what will be outlined in the next subsection.

XI.3.2.3. *Cauchy type formulas and inversion of Radon transform.* Let ω be a planar domain and u be an H^2 -valued function. Then analogs of the standard considerations lead to a Cauchy–Green type formula:

THEOREM 11.2. *If $u \in C^1(\bar{\omega}, H^2)$ and $z_0 \in \omega$, then*

$$(11.20) \quad u(z_0) = \frac{1}{2\pi i} \left\{ \int_{\partial\omega} (dz - L^2 d\bar{z}) (z - z_0 - L^2(\bar{z} - \bar{z}_0))^{-1} u(z) \right\} \\ + \int_{\omega} \int_{\omega} (z - z_0 - L^2(\bar{z} - \bar{z}_0))^{-1} Du(z) dz \wedge d\bar{z}.$$

COROLLARY 11.3. *If $Du = 0$ (i.e., u is A -analytic), then*

$$(11.21) \quad u(z_0) = \frac{1}{2\pi i} \int_{\partial\omega} (dz - L^2 d\bar{z}) (z - z_0 - L^2(\bar{z} - \bar{z}_0))^{-1} u(z)$$

This gives an alternative reconstruction procedure for the standard Radon transform:

- (1) Find the boundary values for u_e and u_o .
- (2) Find u_e and u_o using (11.21).
- (3) Find f from v_o using (11.17).

REMARK 11.4.

- (1) *As one can read in [8, 25], the resolvent $(z - z_0 - L^2(\bar{z} - \bar{z}_0))^{-1}$ involved in the above formulas is formally not defined, since the point $\lambda = (z - z_0)/(\bar{z} - \bar{z}_0)$ always belongs to the spectrum of the operator L^2 . It, however, can be extended from the exterior of the disk $|\lambda| \leq 1$ as a strong limit and hence defined as an unbounded operator.*
- (2) *It is interesting to recognize the standard parts of the Radon inversion formula (Hilbert transform, differentiation, and back projection) in the procedure above. The continuation of the resolvent up to the spectrum gives the Hilbert transform, the integration in the Cauchy formula (11.21) corresponds to the back projection, and the differentiation is performed in (11.17).*

XI.3.2.4. *Inversion of the attenuated transform.* We have dealt so far with the case of zero attenuation. Let us outline now how one can resolve the situation with the attenuated transform, i.e., when the transport equation has the form

$$Du + \mu u = f.$$

Availability of a Cauchy type formula would have the same effect of producing uniqueness result and inversion formulas as above. On a formal level, if we had a “function” h such that

$$(11.22) \quad Dh - \mu h = 0,$$

then Leibnitz’s rule would imply that $D(hu) = hf$. This would mean that the attenuation is essentially eliminated, and the above Cauchy type procedure is applicable to recover hf and hence f . It is easy to find one such function:

$$h(x, \omega) = e^{\mathcal{D}\mu} = e^{\int_0^\infty \mu(x+t\omega)dt}.$$

This, however, is not as simple as it sounds. Indeed, $h(x, \omega)$ for a fixed x acts as the Toeplitz operator $\phi \rightarrow P(h\phi)$ in the Hardy space, and hence we are dealing with an operator valued function of x . On the other hand, the differential operator $D = \bar{\partial} + L^2\partial$ also has operator coefficients. In particular, D and h do not commute in general. This means that the Leibnitz’s rule does not work. If, on the other hand, one could find a solution of (11.22) that for each x has only nonpositive frequencies in its Fourier series with respect to ω , then h and D would commute and the procedure would work. This was implemented (in a not entirely explicit form) in [8], which produced the long awaited proof of uniqueness for the attenuated Radon transform. However, applying more effort one can show that the procedure can be made explicit [95], which leads to a variety of inversion formulas, including the one obtained in a different way in [128] (see also a different derivation in [125]):

$$(11.23) \quad f(x) = -\frac{1}{4\pi} \operatorname{Re} \operatorname{div} \int_{S^1} \omega e^{(\mathcal{D}\mu)(x, \omega^\perp)} (e^{-h} H e^h T_\mu f)(\omega, x\omega) d\omega,$$

where

$$h = \frac{1}{2}(Id + iH)R\mu.$$

This formula was implemented numerically in [58, 96, 98, 125].

XI.3.3. Range conditions. We start with the case of the exponential transform, which is simpler than the one of the general attenuated transform and also leads to interesting analysis.

XI.3.3.1. *Exponential transform.* Probably the first appearance of the range conditions for this transform was in the papers [12] and [167] devoted to its inversion. Consider a compactly supported smooth function $f(x)$ in the plane and its exponential X-ray (Radon) transform with attenuation μ :

$$g(s, \omega, \mu) = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp) e^{\mu t} dt.$$

Here $\omega = (\cos \phi, \sin \phi)$ is a unit vector and $\omega^\perp = (-\sin \phi, \cos \phi)$. Now expand $g(s, \omega, \mu)$ into the Fourier series with respect to the angle ϕ :

$$g(s, \omega, \mu) = \sum_l g_l(s, \mu) e^{il\phi}.$$

Consider the Fourier transform $\widehat{g}_l(\sigma, \mu)$ of $g_l(s, \mu)$ with respect to s . It was observed in [12], [167] that the function

$$(\sigma + \mu)^l \widehat{g}_l(\sigma, \mu)$$

is even with respect to σ for any $l \in \mathbf{Z}$. Papers [88] and [89] were devoted to finding the complete set of range conditions. It was shown in particular that the set of conditions described above is complete. Namely, the following theorem holds:

THEOREM 11.5. *A function $g(s, \omega)$ can be represented as $R_\mu f$ for some $f \in C_0^\infty(\mathbf{R}^2)$ if and only if*

- i) $g \in C_0^\infty(\mathbf{R} \times \mathbf{S}^1)$*
- ii) function $(\sigma + \mu)^l \widehat{g}_l(\sigma, \mu)$ is even with respect to σ for any $l \in \mathbf{Z}$.*

An interesting corollary of the range condition provided in this theorem is that the function $\widehat{g}_l(\sigma, \mu)$ has zero of order $|l|$ at the point $\mu \times \text{sgn } l$. Existence of these zeros, however, is obviously not equivalent to the whole set of conditions. One can check directly that (11.30) in the case of constant attenuation lead to the above mentioned root conditions. This shows in particular that the range conditions (11.30) for the attenuated transform that will be discussed later are incomplete [88].

The range conditions of Theorem 11.5 do not have the usual momentum form as they do in the case of the Radon transform. A momentum type set of conditions was also found in [88] and [89].

THEOREM 11.6. *A function $g(s, \omega)$ can be represented as $R_\mu f$ for some $f \in C_0^\infty(\mathbf{R}^2)$ if and only if*

- i) $g \in C_0^\infty(\mathbf{R} \times \mathbf{S}^1)$*
- ii) the following identity is satisfied for any odd natural n :*

$$(11.24) \quad \sum_{k=0}^n \binom{n}{k} \frac{d}{d\phi} \circ \left(\frac{d}{d\phi} - i \right) \circ \dots \circ \left(\frac{d}{d\phi} - (k-1)i \right) \int_{-\infty}^{\infty} (\mu s)^{n-k} g(s, \omega) ds = 0.$$

Here i is the imaginary unit, $\omega = (\cos \phi, \sin \phi)$, and \circ denotes composition of differential operators.

The condition (11.24) is not very intuitive. One way one can try to understand it is the following. Let us assume that g is representable as $R_\mu f$ and plug $R_\mu f$ instead of g into (11.24). In the case of the standard Radon transform, this would lead to a condition that is obviously satisfied. In other words, for the Radon transform, necessity of the momentum conditions (as soon as they are formulated) is simple to observe, and only sufficiency requires a proof. However, even necessity of (11.24) is not obvious. A direct calculation shows that necessity of this condition is equivalent to the following series of identities for the function $\sin \phi$:

For any odd natural n

$$(11.25) \quad \sum_{k=0}^n \binom{n}{k} \left(\frac{d}{d\phi} - \sin \phi \right) \circ \left(\frac{d}{d\phi} - \sin \phi + i \right) \circ \dots \circ \left(\frac{d}{d\phi} - \sin \phi + (k-1)i \right) \sin^{n-k} \phi = 0.$$

The reader might want to try to establish these identities directly. This was done in [88], where it was also shown that an analogous identity can be written in any commutative differential algebra. A similar identity for all natural numbers n holds for e^x . Further discussion and reformulation of these conditions can be found in [131].

A better understanding of the meaning of the range conditions for the exponential Radon transform can be achieved using complex analysis. This was done in the series of papers [3, 4, 131, 132]. It was shown that the range description problem essentially reduces to Bernstein-Hartogs' type theorems on extension of separately analytic functions (see the details later on in this chapter).

Consider again formula (11.12):

$$\widehat{R_\mu f}(\sigma, \omega) = \sqrt{2\pi} \widetilde{f}(\sigma\omega + i\mu\omega^\perp).$$

If now we have a function $g(s, \omega)$ which we suspect of being in the range of the operator R_μ , then the function defined on S_μ as

$$(11.26) \quad F(\sigma\omega + i\mu\omega^\perp) = \widehat{g}(\sigma, \omega)$$

must be extendable from S_μ to the whole \mathbf{C}^2 as an entire function of the Paley-Wiener class. If this is correct, then the extension provides the Fourier transform of a compactly supported function f such that $g = R_\mu f$. So, the question arises of extendability of the function defined in the left hand side of (11.26) to an entire function of the Paley-Wiener class. Such an extendability condition would provide a range description for the operator R_μ . In fact, such a condition can be easily found and has a simple geometric meaning.

Returning to the projection-slice formula (11.12), we notice that although we initially used the real values of σ only, we can in fact use all complex values of σ , provided that the function f is compactly supported. This leads to determination of values of \widetilde{f} on a three-dimensional variety in \mathbf{C}^2 . Let us denote this variety by Σ_μ :

$$\Sigma_\mu = \{z = \sigma\omega + i\mu\omega^\perp \mid \sigma \in \mathbf{C}, \omega \in \mathbf{S}^1\} \subset \mathbf{C}^2.$$

This variety is smooth at all points where $Re \sigma \neq 0$. At these points it has a complex tangent line. Thus, a boundary $\bar{\partial}$ equation must be satisfied if we want to have our function to be extendable. One can check, however, that this condition is automatically satisfied due to analyticity of $\widehat{g}(\sigma, \omega)$ with respect to σ . So, where is the extendability condition hidden? It turns out that the right place to look for it is at imaginary values of σ . If $\sigma = ip$ for a real p , then the definition of the function F looks as follows:

$$(11.27) \quad F(i(p\omega + \mu\omega^\perp)) = \widehat{g}(ip, \omega).$$

The line $p \rightarrow p\omega + \mu\omega^\perp$ is tangent at the point $\mu\omega^\perp$ to the circle of radius μ in the plane. Changing ω , we rotate this tangent line, spanning the whole exterior of the disk of radius μ . Let us also notice that the function F is entire with respect to p (i.e., along each tangent line). Since there are two such tangent lines passing through each point in the exterior of this disk, the formula (11.27) might provide self-contradictory definition of the function F . Let us impose condition that guarantees that this does not happen. Namely, we assume that

$$(11.28) \quad \widehat{g}(ip_1, \omega_1) = \widehat{g}(ip_2, \omega_2)$$

whenever

$$p_1\omega_1 + \mu\omega_1^\perp = p_2\omega_2 + \mu\omega_2^\perp.$$

It turns out that this simple geometric condition is sufficient. Namely, the following theorem holds.

THEOREM 11.7. *Let F be a function defined in the exterior of the disk of a positive radius in the plane. If this function is entire along each tangent line to the disk, it is extendable to an entire function of two variables.*

This theorem was proven (although not formulated explicitly in the present form) in [3]. The proof uses an analytic trick, which role has not yet been well understood. For instance, the question remains whether one can replace tangent lines to a circle by tangent lines to a more general convex algebraic curve. This is certainly a separate analyticity theorem of S. Bernstein's kind [6, 18], since in a neighborhood of any point in the exterior of the disk the pairs of tangent lines provide a local coordinate system, for which we have separate analyticity. A proof that uses purely complex analytic tools was provided in [132]. It was shown in [136] and [131] that the strange moment conditions (11.24) can be understood as the infinitesimal form of (11.28) at the boundary of the disk.

The last theorem leads to the following range description [3].

THEOREM 11.8. *Let $g(s, \omega)$ be a function on $\mathbf{R} \times S^1$, and μ be a positive real. Then $g = R_\mu f$ for some function $f \in C_0^\infty(\mathbf{R}^2)$ if and only if $g \in C_0^\infty(\mathbf{R} \times \mathbf{S}^1)$, and the Fourier transform $\widehat{g}(\xi, \phi)$ with respect to s -variable of $g(s, \omega(\phi))$ satisfies the following condition for all real σ :*

$$(11.29) \quad \widehat{g}(i\sigma, \phi - \arcsin \frac{\sigma}{\sqrt{\sigma^2 + \mu^2}}) = \widehat{g}(-i\sigma, \phi + \arcsin \frac{\sigma}{\sqrt{\sigma^2 + \mu^2}}).$$

Here $\omega(\phi) = (\cos \phi, \sin \phi)$.

The condition (11.29) is a simple restatement of (11.28).

A generalization of this result to the multidimensional case is provided in [4]. A nice discussion can be found in [131] and [132].

Let us mention briefly some applications of these range conditions. It was shown in [137] how the conditions can be effectively used in SPECT in order to detect and correct some data errors arising from hardware imperfection. The paper [111] uses the range conditions in order to treat numerically incomplete data problems of SPECT. It is suggested in paper [32]

that the range conditions might be used in some problems of radiation treatment planning (see also [31] and [90]). We will not go into details of these results.

XI.3.3.2. *Attenuated transform.* The question about the range descriptions for the attenuated Radon transform turns out to be a tricky one. In fact, one could think that there is no chance to get any explicit range conditions for the attenuated X-ray transform. Despite this natural belief, F. Natterer [124] found the following infinite set of such range conditions. Let us introduce some notation first. We denote by H the Hilbert transform on the line:

$$Hp(x) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{p(y)}{x-y} dy$$

where $v.p.$ denotes the principal value integral, (the limit as $\epsilon \rightarrow 0$ of the integral over $\mathbf{R} \setminus (-\epsilon, \epsilon)$).

THEOREM 11.9. [124] *Let f and μ belong to the Schwartz space $\mathcal{S}(\mathbf{R}^2)$. Then, for $k > m \geq 0$ integers, we have*

$$(11.30) \quad \int_{-\infty}^{\infty} \int_0^{2\pi} s^m e^{\pm ik\phi + 0.5(I \pm iH)R\mu(\omega, s)} T_{\mu} f(\omega, s) d\phi ds = 0,$$

where $\omega = (\cos \phi, \sin \phi)$, I is the identity operator, H is the Hilbert transform, and $R\mu$ is the Radon transform of μ .

As has already been explained, the set of conditions provided in this theorem is incomplete. Nevertheless, it is of practical importance, since it can be used for the simultaneous recovery of the sources $f(x)$ and attenuation $\mu(x)$ (see [121]-[123] and further discussion below).

The question had remained on how to write down the missing conditions, until the very recent publication [130], where a complete set of conditions was obtained.

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It can be shown ([95]) that the approach of [8] provides an explanation and natural derivation of (11.30). In order to explain this, let us start with the case of zero attenuation (i.e., standard Radon transform) and look at it in terms of A -analytic functions (see Section XI.3.2.2). In the notation of that section, the role of the Radon data is played by a function u on $\partial\Omega$ with values in the Hardy space. It belongs to the range **iff** it is A -analytically extendable to Ω . Due to an analog of Sokhotsky's formula for such functions, it is easy to show that this happens **iff**

$$(11.31) \quad \int_{\partial\omega} (dz - L^2 d\bar{z}) (z - z_0 - L^2 (\bar{z} - \bar{z}_0))^{-1} u(z) = 0$$

for any $z_0 \notin \bar{\Omega}$. These conditions, however, do not resemble the standard momentum ones. How do we get those? In standard complex analysis, the boundary values of an analytic function as a measure on $\partial\Omega$ must be orthogonal to the boundary values of any function analytic in Ω and continuous up to the boundary. Any base of such functions would do. For

instance, powers of z give

$$\int_{S^1} z^k u(z) dz = 0, \quad k = 0, 1, 2, \dots$$

Similarly for A -analytic functions: (11.31) means orthogonality on the boundary to the operator functions

$$L_{z_0}(z) = (z - z_0 - L^2(\bar{z} - \bar{z}_0))^{-1}.$$

Is there any other convenient basis? What are the analogs of the powers z^n here? The operator valued function $H(z) = Rz - L\bar{z}$ satisfies $DH = 0$. It is an A -analytic analog of z . Due to the operator nature of H and D , the product Hu is not necessarily A -analytic. Namely,

$$D(Hu) = \left(\frac{\partial}{\partial \bar{z}} + L^2 \frac{\partial}{\partial z} \right) (Rz - L\bar{z}) u(z) = [L^2, R] \frac{\partial u}{\partial z} z.$$

This expression is not necessarily equal to zero. However, the commutator $[L^2, R]$ equals $-P_1 L$, where P_l is the projector onto the first l terms of Fourier expansion. In particular, $L[L^2, R] = 0$. This implies that

$$L \int_{\partial\omega} (dz - L^2 d\bar{z}) Hu = 0$$

and analogously

$$L^k \int_{\partial\omega} (dz - L^2 d\bar{z}) H^k u = 0, \quad k = 0, 1, \dots$$

THEOREM 11.10. [95] *These conditions coincide (modulo evenness) with the standard momentum conditions.*

How can one now include attenuation into this scheme? Using an appropriate “correction” function h as in Section XI.3.2.2 and applying then the above conditions to hu one obtains the range conditions. In particular, one can obtain (11.30).

THEOREM 11.11. [95] *Orthogonality to the suitably understood functions $H^k h$ (i.e., analogs of orthogonality to z^k) gives Natterer’s range conditions (11.30).*

We would like to mention again that a full set of range conditions was obtained in [130].

XI.3.4. Recovery of attenuation. As we have discussed in the beginning of this section, simultaneous recovery of the sources density $f(x)$ and of the attenuation $\mu(x)$ is an important applied issue. At the first glance this problem might look hopeless. Indeed, in the plane case the data $g = T_\mu f$ is a single function of two variables, while we are trying to recover two functions $f(x)$ and $\mu(x)$ of two variables. However, this counting of functions and variables would be persuasive only if the operator T_μ were close to an invertible one (for instance, if it were a Fredholm operator). In fact, the study of the range conditions shows that the operator T_μ has an infinite dimensional cokernel. This means that when μ changes, the range could in principle rotate in such a way that for two different values of the attenuation the two ranges would have zero (or a “very small”) intersection. If this were

true, then both f and μ would be recoverable or “almost recoverable”. For instance, some additional *a priori* information could help. We will describe now some results that show that something like this does happen, although the understanding of the situation is still very superficial.

Let us start with the simplest case of the exponential X-ray transform. The first attempt to recover both f and μ from $R_\mu f$ was made in [75]. This problem was resolved in [162] (see also [163]). Consider the range conditions in the form of evenness of the function

$$(\sigma + \mu)^l \widehat{g}_l(\sigma, \mu).$$

This implies the equality

$$\left(\frac{\sigma + \mu}{-\sigma + \mu} \right)^l = \frac{\widehat{g}_l(-\sigma, \mu)}{\widehat{g}_l(\sigma, \mu)},$$

whenever $\widehat{g}_l(\sigma, \mu)$ is not identically zero. This means that if there is a nonzero harmonic $\widehat{g}_l(\sigma, \mu)$ for some $l \neq 0$, then the rational function

$$\frac{\sigma + \mu}{-\sigma + \mu}$$

is uniquely determined. This, in turn, determines μ uniquely as the pole of this function. The only exception arises when $\widehat{g}_l(\sigma, \mu) = 0$ for all $l \neq 0$. In this case, however, the function $g = R_\mu f$, and hence f itself, is radial. For radial functions no information about the value of μ can be extracted from $R_\mu f$. Geometrically this means that the intersection of ranges of R_μ for different values of μ consists of radial functions. In other words, we established the following theorem.

THEOREM 11.12. [162, 163] *Simultaneous recovery of a compactly supported function f and constant attenuation μ from the values of $R_\mu f$ is possible if and only if f is not radial.*

In fact, in the non-radial case one can provide an algorithm of finding the attenuation μ (see [162] and [163]).

An alternative approach to recovery of attenuation was described in [7].

Recovery of a variable attenuation is definitely much more difficult. As in the exponential case, the range theorems are used to this end. Theorem 11.9 has been used under some additional restrictions for the sources distribution in order to accomplish this task [121, 122, 123]. Papers [8, 128, 129, 130] contain some additional indications on how their techniques and results might be employed for that purpose.

XI.4. Hyperbolic Integral Geometry and Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) is a very promising, inexpensive, and technologically simple (at least in comparison with X-ray, SPECT, PET, and MRI scans) method of medical diagnostics and of industrial nondestructive testing (see, for instance [9, 10, 16, 17, 23, 28, 29, 47, 54, 77, 87, 104, 150, 154, 155, 164, 165] and references therein). Here is the idea of EIT: one places electrodes around a body, creates some known currents through the electrodes, and then measures the corresponding boundary voltage drops. All

of the data are collected on the boundary. After many such measurements one wants to recover the distribution of the electric conductivity inside the body. The information about the electric conductivity is very important for medical diagnostics; it is also vital for some electrical procedures, such as defibrillation; it could also provide a cheap and reliable non-destructive evaluation technology. There are several strong groups in industry and academy that work on the practical and theoretical aspects of this problem using various techniques. We will consider here only one approach, which has relation to the integral geometry problems discussed in the book. The reader can refer to the papers quoted above that address other methods that do not use Radon transform.

Let us describe first the set up of the inverse conductivity problem, which is the mathematical formulation of EIT. Let $U \subset \mathbf{R}^n$ be a domain with sufficiently smooth boundary Γ . An unknown function $\beta(x)$ (the conductivity) must be recovered from the following data. Given a function ψ on Γ (the current) one solves the Neumann boundary value problem

$$\begin{cases} \nabla \cdot (\beta \nabla u) = 0 & \text{in } U \\ \beta \frac{\partial u}{\partial \nu} \Big|_{\Gamma} = \psi, \end{cases}$$

where ν is the unit outer normal vector on Γ . Then one measures the boundary value $\phi = u|_{\Gamma}$ (the potential). All the pairs (ψ, ϕ) are assumed to be accessible. In other words, the so called Dirichlet-to-Neumann operator $\Lambda_{\beta} : \phi \rightarrow \psi$ is known. One needs to recover the conductivity β from this data. The problem is obviously nonlinear.

The inverse conductivity problem is a hard nut to crack both analytically and numerically. The main questions are about uniqueness of determination of the conductivity, stability of the reconstruction, and inversion algorithms. Although the problem of uniqueness can be considered as principally resolved (see [119, 166, 170, 78], and references therein), the other ones are still under thorough investigation. The general understanding is that the problem is highly unstable, so there is no hope to achieve the quality of reconstruction known to other common tomographic techniques, at least without involving additional information about the image to be reconstructed.

The first practical algorithm of D. Barber and B. Brown [9]-[11] deals with the linearized problem, which is still highly unstable. A thorough investigation of this algorithm started by F. Santosa and M. Vogelius [155] led in works of C. Berenstein and E. Casadio Tarabusi [16, 17] to the understanding that the linearized two-dimensional problem can be treated by means of hyperbolic geometry. Consider the two-dimensional case when U is the unit disk. It is well known (see for instance [13, 69]) that the unit disk serves as the Poincaré model of the hyperbolic plane \mathbf{H}^2 . Here are the indications why the hyperbolic geometry might play some role in the (at least linearized) inverse conductivity problem. First of all, the Laplace operator that arises in the linearized problem is invariant with respect to the group of Möbius transformations. Another indication is that if one creates a dipole current through a point on the boundary of U , then the equipotential lines and the current lines form families of geodesics and horocycles in \mathbf{H}^2 . Following the analysis done in [155] of the algorithm suggested in [9, 10], C. Berenstein and E. Casadio Tarabusi ([16, 17]) discovered that in fact for $n = 2$ the linearized inverse conductivity problem can be reduced to the following

integral geometry problem on \mathbf{H}^2 : the available data enables one to find the function

$$R_G(A * \beta),$$

where R_G is the geodesic Radon transform on \mathbf{H}^2 , A is an explicitly described radial function on \mathbf{H}^2

$$A(r) = \text{const}(3 \cosh^{-4} r - \cosh^{-2} r),$$

and the star $*$ denotes the (non-Euclidean) convolution on \mathbf{H}^2 . Now one can hope to use the known methods of harmonic analysis on \mathbf{H}^2 to recover β . Namely, one needs to be able to invert the geodesic Radon transform and to deconvolve. Necessary details of harmonic analysis on \mathbf{H}^2 can be found in [68, 70]. Several different inversion formulas for the geodesic Radon transform are developed in [69, 15, 106]. The paper [53] provides a unified approach to such formulas. The reader might also be interested in the related papers [99, 101]. The formula obtained in [106] was numerically implemented in [45]. Another part of inversion would be deconvolution. Its numerical implementation can be done by using the Fourier transform on the hyperbolic plane described in [68] and its inversion. The Fourier transform is defined as follows:

$$f(z) \rightarrow Ff(\lambda, b) = \int_{\mathbf{H}} f(z) e^{(-i\lambda+1)\langle z, b \rangle} dm(z),$$

where $b \in \partial\mathbf{H}^2$, $\lambda \in \mathbf{R}$, and $\langle z, b \rangle$ is the (signed) hyperbolic distance from the origin to the horocycle passing through the points z and b , and $dm(z)$ is the invariant measure on \mathbf{H}^2 . The inverse Fourier transform is

$$g(\lambda, b) \rightarrow F^{-1}g(z) = \text{const} \int \int g(\lambda, b) e^{(i\lambda+1)\langle z, b \rangle} \lambda \tanh(\pi\lambda/2) d\lambda db.$$

These operators were numerically implemented in [45]. (By an editorial error, all pictures have been omitted in [45]. They can be found at the URL <http://www.math.tamu.edu/kuchment/hypnum.pdf>.) Although the problem of numerical implementation of all these transforms looks familiar and similar to the Euclidean case, it is in fact much more complex. In particular, even the problem of sampling is not a trivial one. One natural way of sampling would be the following. One chooses a discontinuous group of motions of the hyperbolic disk and uses the orbit of the origin as the grid. Choosing a small mesh, one samples with appropriate accuracy. Unfortunately, this idea does not work, due to the known rigidity of such lattices. Namely, the size of the pixel (i.e., the fundamental domain) for any discrete group acting on \mathbf{H}^2 cannot be made arbitrarily small [13]. The way of overcoming this difficulty chosen in [45] was to create a combination of Euclidean and hyperbolic grids. A “superlattice” was created using a discrete group of hyperbolic motions, and then a finer Euclidean grid was chosen in each of the fundamental domains.

It seemed at first that the hyperbolic geometry approach would not work in dimensions higher than two, due to lack of hyperbolic invariance of the governing equations. It was shown, however, in [46] that a combination of Euclidean and hyperbolic integral geometries does the trick.

Bibliography

- [1] M. Agranovsky, C.A. Berenstein, and P. Kuchment, Approximation by spherical waves in L^p -spaces, *J. Geom. Analysis*, 6(1996), no. 3, 365–383
- [2] M. Agranovsky and E.T. Quinto, Injectivity sets for Radon transform over circles and complete systems of radial functions, *J. Functional Anal.*, 139(1996), 383–414.
- [3] V. Aguilar, L. Ehrenpreis, and P. Kuchment, Range conditions for the exponential Radon transform, *J. d’Analyse Mathématique*, 68(1996), 1-13.
- [4] V. Aguilar and P. Kuchment, Range conditions for the multidimensional exponential X-ray transform, *Inverse Problems* 11(1995), 977-982.
- [5] M. D. Altschuler, Reconstruction of the global-scale three-dimensional solar corona, in *Image Reconstruction from Projections, Implementation And Applications*, Ed. G. T. Herman, Topics in Applied Physics 32, 105–145, Springer-Verlag, New York/Berlin, 1979.
- [6] N. I. Akhiezer and L. I. Ronkin, On separately analytic functions of several variables and theorems on “the thin edge of the wedge”, *Russian Math. Surveys*, 28(1973), 27-42.
- [7] Yu. E. Anikonov and I. Shneiberg, Radon transform with variable attenuation, *Doklady Akad. Nauk SSSR*, 316(1991), no.1, 93-95. English translation in *Soviet. Math. Dokl.*
- [8] E.V. Arbuzov, A.L. Bukhgeim, and S.G. Kasantzev, Two-dimensional tomography problems and the theory of A -analytic functions, *Siberian Adv. in Math.* 8(1998), no.4, 1-20.
- [9] D.C. Barber, B.H. Brown, Applied potential tomography, *J. Phys. E.: Sci. Instrum.* 17(1984), 723-733.
- [10] D.C. Barber, B.H. Brown, Recent developments in applied potential tomography-APT, in: *Information Processing in Medical Imaging*, Nijhoff, Amsterdam, 1986, 106-121.
- [11] D.C. Barber, B.H. Brown, Progress in electrical impedance tomography, in *Inverse Problems in Partial Differential Equations*, SIAM, 1990, pp. 151-164.
- [12] S. Bellini, M. Piacentini, C. Caffario, and P. Rocca, Compensation of tissue absorption in emission tomography, *IEEE Trans. ASSP* 27(1979), 213-218.
- [13] A. Berdon, *The Geometry of Discrete Groups*, Springer Verlag, New York- Heidelberg-Berlin, 1983.
- [14] C. Berenstein, Radon transforms, wavelets, and applications, in *Integral Geometry, Radon Transforms and Complex Analysis (Venice, 1996)*, 1–33, Lecture Notes in Math., 1684, Springer, Berlin, 1998.
- [15] C. Berenstein, E. Casadio Tarabusi, Inversion formulas for the k -dimensional Radon transform in real hyperbolic spaces, *Duke Math. J.* 62(1991),no.3, 613- 631.
- [16] C. Berenstein, E. Casadio Tarabusi, The inverse conductivity problem and the hyperbolic X-ray transform, pp. 39-44 in [52].
- [17] C. Berenstein, E. Casadio Tarabusi, Integral geometry in hyperbolic spaces and electrical impedance tomography, *SIAM J. Appl. Math.* 56(1996), 755-764.
- [18] S. Bernstein, Sur l’ordre de la meilleure approximation des fonctions continues par des polynomes de degre donne, *Académie Royale de Belgique, Classe des Sciences, Mémoires Collection*, 4(1922), 1-103.
- [19] G. Beylkin, The inversion problem and applications of the generalized Radon transform, *Comm. Pure Appl. Math.* 37(1984), 579–599.
- [20] J. Boman, An example of nonuniqueness for a generalized Radon transform. *J. Anal. Math.* 61(1993), 395–401.

- [21] J. Boman and E. T. Quinto, Support theorems for real analytic Radon transforms, *Duke Math. J.* 55(1987), no.4, 943-948.
- [22] J. Boman and E. T. Quinto, Support theorems for real analytic Radon transforms on line complexes, *Trans. Amer. Math. Soc.* 335(1993), 877-890.
- [23] L. Borcea, J. Berryman, and G. Papanicolaou, High-contrast impedance tomography, *Inverse Problems* 12(1996), 835-858.
- [24] T.F. Budinger, G.T. Gullberg, and R.H. Huseman, Emission computed tomography, pp. 147-246 in [72].
- [25] A.L. Bukhgeim, Inversion formulas in inverse problems, a supplement to M. M. Lavrent'ev and L. Ya. Savel'ev, *Linear Operators and Ill-posed Problems*, Translated from the Russian, Consultants Bureau, New York; "Nauka", Moscow, 1995.
- [26] A.L. Bukhgeim and S. G. Kazantsev, The attenuated Radon transform inversion formula for divergent beam geometry, preprint, 2002.
- [27] E. J. Candès and D. L. Donoho . Curvelets and Reconstruction of Images from Noisy Radon Data, *Wavelet Applications in Signal and Image Processing VIII*, A. Aldroubi, A. F. Laine, M. A. Unser eds., Proc. SPIE 4119.
- [28] M. Cheney, D. Isaacson, and J.C. Newell, Electrical Impedance Tomography, *SIAM Review*, 41, No. 1, (1999), 85-101.
- [29] B. Cipra, Shocking images from RPI, *SIAM News*, July 1994, 14-15.
- [30] A. M. Cormack, Representation of a function by its line integrals with some radiological applications, *J. Appl. Phys.* 34(1963), 2722-2727.
- [31] A. Cormack and E.T. Quinto, A problem in radiotherapy: questions of non-negativity, *Internat. J. Imaging Systems and Technology*, 1(1989), 120-124.
- [32] A. Cormack and E.T. Quinto, The mathematics and physics of radiation dose planning, *Contemporary Math.* 113(1990), 41-55.
- [33] R.A. Crowther, D.J. DeRosier, and A. Klug, The reconstruction of a three-dimensional structure from projections and its application to electron microscopy, *Proc. R. Soc. London, Ser. A.* 317(1962), 319-340.
- [34] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume II Partial Differential Equations*, Interscience, New York, 1962.
- [35] M. Davison, The ill-conditioned nature of the limited angle tomography problem, *SIAM J. Appl. Math.* 43(1983), 428-448.
- [36] J. De Stefano and T. Olsen, Wavelet localization of the Radon transform in even dimensions, *IEEE Trans. Signal Proc.* 42(1994), 2055-2067.
- [37] L. Ehrenpreis, P. Kuchment, and A. Panchenko, The exponential X-ray transform and Fritz John's equation. I. Range description, in *Analysis, Geometry, Number Theory: the Mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998)*, 173-188, *Contemporary Math.*, 251, Amer. Math. Soc., Providence, RI, 2000.
- [38] A. Faridani, E. L. Ritman, and K. T. Smith, Local tomography, *SIAM J. Appl. Math.* 52(1992), 459-484, Examples of Local Tomography, *SIAM J. Appl. Math.* 52(1992), 1193-1198.
- [39] A. Faridani, D.V. Finch, E.L. Ritman, and K. Smith, Local Tomography II, *SIAM J. Appl. Math.* 57(1997), 1095-1127.
- [40] D.V. Finch, Cone beam reconstruction with sources on a curve, *SIAM J. Appl. Math.* 45(1985), 665-673.
- [41] D.V. Finch, Uniqueness for the X-ray transform in the physical range, *Inverse Problems* 2(1986), 197-203.
- [42] D.V. Finch, Talk at Mt. Holyoke AMS/IMS/SIAM summer conference, June 18-22, 2000.

- [43] D.V. Finch, The attenuated X -ray transform: recent developments, to appear in *Inside - Out: Inverse Problems*, G. Uhlmann (Editor), Cambridge Univ. Press.
- [44] D.V. Finch and A. Hertle, The exponential Radon transform, *Contemporary Math.* 63(1987), 67-74.
- [45] B. Fridman, P. Kuchment, K. Lancaster, S. Lissianoi, D. Ma, M. Mogilevsky, V. Papanicolaou, and I. Ponomarev, Numerical harmonic analysis on the hyperbolic plane. *Appl. Anal.* 76(2000), no. 3-4, 351-362.
- [46] B. Fridman, P. Kuchment, D. Ma, and V. Papanicolaou, Solution of the linearized conductivity problem in the half space via integral geometry, in *Voronezh Winter Mathematical Schools*, P. Kuchment (Editor), 85-95, Amer. Math. Soc. Transl. Ser. 2, 184, Amer. Math. Soc., Providence, RI, 1998.
- [47] A. Friedman, M. Vogelius, Determining cracks by boundary measurements, *Ind. Univ. Math. J.* 38(1989), 527-556.
- [48] I. Gelfand and S. Gindikin (Editors), *Mathematical Problems of Tomography*, AMS, Providence, R.I. 1990.
- [49] I. Gelfand, S. Gindikin, and M. Graev, Integral geometry in affine and projective spaces, *J. Sov. Math.* 18(1980), 39-167.
- [50] I. Gelfand, M. Graev, and N. Vilenkin, *Generalized Functions, Vol.5: Integral Geometry and Representation Theory*, Acad. Press, New York 1966.
- [51] S. Gindikin (Editor), *Applied Problems of the Radon Transform*, AMS, Providence, RI, 1994.
- [52] S. Gindikin and P. Michor (Editors), *75 Years of the Radon Transform*, Internat. Press 1994
- [53] S. Gindikin, Integral geometry on real quadrics, in *Lie groups and Lie algebras: E. B. Dynkin's Seminar*, 23-31, Amer. Math. Soc. Transl. Ser. 2, 169, Amer. Math. Soc., Providence, RI, 1995.
- [54] D. Gisser, D. Isaacson, J. Newell, Current Topics in Impedance Imaging, *Clin.Phys.Physiol. Meas.*, 8(1987), 216-241.
- [55] J. Globevnik, Partial indices of analytic discs attached to Lagrangian submanifolds of C^N , *Annales de l'Institut Fourier* 46(1996), fasc. 5, p. 1305-1324.
- [56] A. Greenleaf and G. Uhlmann, Non-local inversion formulas for the X-ray transform, *Duke Math. J.*, 58(1989), 205-240.
- [57] A. Greenleaf and G. Uhlmann, Microlocal techniques in integral geometry, *Contemporary Math.* 113(1990), 149-155.
- [58] J.-P. Guillemin, F. Jauberteau, L. Kunyansky, R. Novikov, and R. Trebossen, On SPECT imaging based on an exact formula for the nonuniform attenuation correction, preprint, 2002.
- [59] V. Guillemin, Fourier integral operators from the Radon transform point of view, *Proc. Symposia in Pure Math.*, 27(1975) 297-300.
- [60] V. Guillemin, Some remarks on Integral geometry, manuscript, 1975.
- [61] V. Guillemin and S. Sternberg *Geometric Asymptotics*, Amer. Math. Soc., Providence, RI, 1977.
- [62] On some results of Gelfand in integral geometry *Proc. Symposia in Pure Math.*, 43(1985) 149-155.
- [63] I. Hazou and D. Solmon, Inversion of the exponential Radon transform I, *Analysis, Math. Methods in Appl. Sci.* 10(1988), 561-574.
- [64] I. Hazou and D. Solmon, Inversion of the exponential Radon transform II, *Numerics, Math. Methods Appl. Sci.* 13(1990), no. 3, 205-218.
- [65] I. Hazou and D. Solmon, Filtered-backprojection and the exponential Radon transform, *Math. Anal. Appl.* 141(1989), no.1, 109-119.
- [66] U. Heike, Single-photon emission computed tomography by inverting the attenuated Radon transform with least-squares collocation, *Inverse Problems* 2(1986), 307-330.
- [67] S. Helgason, *The Radon Transform*, Birkhäuser, Basel 1980.
- [68] S. Helgason, *Groups and Geometric Analysis*, Acad. Press, NY 1984.
- [69] S. Helgason, The totally-geodesic Radon transform on constant curvature spaces, *Contemporary Math.* 113(1990), 141-149.

- [70] S. Helgason, *Geometric Analysis on Symmetric Spaces*, AMS, Providence, RI 1994.
- [71] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassman manifolds, *Acta Math.* 113(1965), 153–180.
- [72] G. Herman (Ed.), *Image Reconstruction from Projections*, Topics in Applied Physics, v. 32, Springer Verlag, Berlin, New York 1979.
- [73] G. Herman, *Image Reconstruction from Projections*, Acad. Press, NY 1980
- [74] A. Hertle, On the injectivity of the attenuated Radon transform, *Proc. Amer. Math. Soc.* 92(1984), 201-205.
- [75] A. Hertle, The identification problem for the constantly attenuated Radon transform, *Math. Z.* 197(1988), 13-19.
- [76] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer Verlag, New York, 1983.
- [77] D. Isaacson and M. Cheney, Current problems in impedance imaging, in *Inverse Problems in Partial Differential Equations*, SIAM, 1990, pp. 141-149.
- [78] V. Isakov, *Inverse Problems for Partial Differential Equations*, Applied Mathematical Sciences, v. 127. Springer-Verlag, New York, 1998.
- [79] F. John, The ultrahyperbolic differential equation with four independent variables, *Duke Math. J.* 4(1938), 300-322.
- [80] A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging*, IEEE Press, 1988.
- [81] A. Katsevich, Local tomography for the generalized Radon transform, *SIAM J. Appl. Math.* 57(1997), 1128–1162.
- [82] A. Katsevich, Local tomography with nonsmooth attenuation, *Trans. Amer. Math. Soc.* 351(1999), no. 5, 1947–1974.
- [83] A. Katsevich, Cone beam local tomography, *SIAM J. Appl. Math.* 59(1999), 2224-2246.
- [84] A. Katsevich and A.G. Ramm, A method for finding discontinuities of functions from the tomographic data, pp. 115-124 in [150].
- [85] A. Katsevich and A.G. Ramm, Pseudolocal Tomography, *SIAM J. Appl. Math.*(1996), 167–191.
- [86] A.I. Katsevich and A.G. Ramm, A method for finding discontinuities from the tomographic data, pp. 115–123 in [150].
- [87] R. Kohn, M. Vogelius, Determining Conductivity by Boundary Measurements, I, *Comm. Pure Appl. Math.*, 37(1984), 289-298; II, 38(1985), 643-667.
- [88] P. Kuchment and S. Lvin, Paley-Wiener theorem for exponential Radon transform, *Acta Appl. Math.* 18(1990), 251-260
- [89] P. Kuchment and S. Lvin, The range of the exponential Radon transform, *Soviet Math. Dokl.* 42(1991), no.1, 183-184.
- [90] P. Kuchment, On positivity problems for the Radon transform and some related transforms, *Contemporary Math.*, 140(1993), 87-95.
- [91] P. Kuchment, On inversion and range characterization of one transform arising in emission tomography, pp. 240-248 in [52].
- [92] P. Kuchment, private communication, 1994.
- [93] P. Kuchment and I. Shneiberg, Some inversion formulas in the single photon emission tomography, *Appl. Anal.* 53(1994), 221-231.
- [94] P. Kuchment, K. Lancaster, and L. Mogilevskaya, On local tomography, *Inverse Problems*, 11(1995), 571-589.
- [95] P. Kuchment, a lecture at the AMS Annual meeting, New Orleans 2001.
- [96] L. Kunyansky, A new SPECT reconstruction algorithm based on the Novikov’s explicit inversion formula, *Inverse Problems* 17(2001), 293–306.

- [97] L. Kunyansky, Inversion of the exponential ray transform in 3 – D using parallel–beam data measured from closed orbits, preprint 2002.
- [98] L. Kunyansky, Analytic reconstruction algorithms in emission tomography with variable attenuation, to appear in *J. Comp. Math. in Sci. and Eng.*
- [99] A. Kurusa, The Radon transform on hyperbolic space, *Geometriae Dedicata* 40(1991), no.2, 325-339.
- [100] A. Kurusa, The invertibility of the Radon transform on abstract rotational manifolds of real type, *Math. Scand.* 70(1992), 112-126.
- [101] A. Kurusa, Support theorems for totally geodesic Radon transforms on constant curvature spaces, *Proc. AMS* 122(1994), no.2, 429-435
- [102] I.-R. Lan, *On an operator associated to a restricted x-ray transform*, Ph.D. Thesis, Oregon State University, 2000.
- [103] R. M. Lewitt and R. H. T. Bates, Image reconstruction from projections. **II**: Projection completion methods (theory), *Optik* 50(1978), 189–204, **III**: Projection completion methods (computational examples) *Optik* 50(1978), 269–278.
- [104] V. Liepa, F. Santosa, and M. Vogelius, Crack determination from boundary measurements - reconstruction from experimental data, *J. Nondestructive Evaluation* 12(1993), 163-173.
- [105] S. Lissianoi, On stability estimates in the exterior problem for the Radon transform, in *Tomography, Impedance Imaging, and Integral Geometry (South Hadley, MA, 1993)*, 143–147, Lectures in Appl. Math., 30, Amer. Math. Soc., Providence, RI, 1994.
- [106] S. Lissianoi and I. Ponomarev, On the inversion of the geodesic Radon transform on the hyperbolic plane, *Inverse Problems* 13(1977), 1-10.
- [107] A. Louis, Incomplete data problems in X-ray computerized tomography 1. Singular value decomposition of the limited angle transform, *Numer. Math.* 48(1986), 251–262.
- [108] A. Louis, Private communication, 1988.
- [109] A.K. Louis and P. Maass, Contour reconstruction in 3-D X-Ray CT, *IEEE Trans. Medical Imaging*, 12(1993), 764-769.
- [110] A. Louis and A. Rieder, Incomplete data problems in X-ray computerized tomography II. Truncated projections and region-of-interest tomography, *Numer. Math.* 56(1989), 371–383.
- [111] S. Lvin, Data correction and restoration in emission tomography, pp. 149–155 in [150].
- [112] D. Ma, Personal communication, 1995.
- [113] P. Maass, 3D Röntgentomographie: Ein Auswahlkriterium für Abtastkurven, *Z. angew. Math. Mech.* 68(1988), T 498–T499.
- [114] P. Maass, The interior Radon transform, *SIAM J. Appl. Math.* 52(1992), 710–724.
- [115] W. Madych, Reconstruction and wavelet transforms for quasilocal tomography, preprint.
- [116] W. Madych and S. Nelson, Reconstruction from restricted Radon transform data: resolution and ill-conditionedness, *SIAM J. Math. Anal.* 17(1986), 1447–1453.
- [117] A. Markoe, Fourier inversion of the attenuated X-ray transform, *SIAM J. Math. Anal.* 15(1984), no.4, 718-722.
- [118] A. Markoe and E. T. Quinto, An elementary proof of local invertibility for generalized and attenuated Radon transforms. *SIAM J. Math. Anal.* 16(1985), no. 5, 1114–1119.
- [119] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. Math.* 143(1996), 71-96.
- [120] F. Natterer, Efficient implementation of “optimal” algorithms in computerized tomography, *Math. Meth. Appl. Sci.* 2(1980), 545–555.
- [121] F. Natterer, On the inversion of the attenuated Radon transform, *Numer. Math.* 32(1979), 431-438.
- [122] F. Natterer, Computerized tomography with unknown sources, *SIAM J. Appl. Math.* 43(1983), 1201-1212.

- [123] F. Natterer, Exploiting the range of Radon transform in tomography, in: Deuffhard P. and Hairer E. (Eds.), Numerical treatment of inverse problems in differential and integral equations, Birkhäuser Verlag, Basel 1983.
- [124] F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986.
- [125] F. Natterer, Inversion of the attenuated Radon transform, *Inverse Problems* 17(2001), no. 1, 113–119.
- [126] F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, Monographs on Mathematical Modeling and Computation v. 5, SIAM, Philadelphia, PA 2001.
- [127] F. Noo and J.–M. Wagner, Image reconstruction in 2D SPECT with 180° acquisition, *Inverse Problems*, 17(2001), 1357–1371.
- [128] R. Novikov, Une formule d’inversion pour la transformation d’un rayonnement X atténue, *C. R. Acad. Sci. Paris Ser. I Math* 332(2001), no. 12, 1059–1063.
- [129] R. Novikov, An inversion formula for the attenuated X -ray transformation, *Ark. Math.*, 40(2002), 145–167.
- [130] R. Novikov, On the range characterization for the two-dimensional attenuated X -ray transform, *Inverse Problems* 18(2002), 677–700.
- [131] O. Öktem, Comparing range characterizations of the exponential Radon transform, *Research reports in Math.*, no.17, 1996, Dept. Math., Stockholm Univ., Sweden.
- [132] O. Öktem, Extension of separately analytic functions and applications to range characterization of the exponential Radon transform, in *Complex Analysis and Applications (Warsaw, 1997)*, *Ann. Polon. Math.* 70(1998), 195–213.
- [133] V. Palamodov, Nonlinear artifacts in tomography *Sov. Phys. Dokl.* 31(1986), 888–890.
- [134] R. M. Perry, On reconstruction of a function on the exterior of a disc from its Radon transform, *J. Math. Anal. Appl.* 59(1977), 324–341.
- [135] B. Petersen, *Introduction to the Fourier Transform and Pseudo-Differential Operators*, Pittman, Boston, 1983.
- [136] I. Ponomarev, On reconstruction of projection data containing strong displacement distortions in emission gamma-tomography, in *Proc. Conf. on Ill-posed Probl. of Math. Phys. and Anal.*, Novosibirsk, 1992, p. 111.
- [137] I. Ponomarev, Correction of emission tomography data. Effects of detector displacement and non-constant sensitivity, *Inverse Problems*, 10(1995) 1-8.
- [138] N.G. Preobrazhensky and V.V. Pikalov, *Unstable Problems of Plasma Diagnostic*, Nauka, Novosibirsk, 1982.
- [139] N.G. Preobrazhensky and V.V. Pikalov, *Reconstructive Tomography in Gas Dynamics and Plasma Physics*, Nauka, Novosibirsk, 1987.
- [140] E. T. Quinto, The dependence of the generalized Radon transform on defining measures, *Trans. Amer. Math. Soc.* 257(1980), 331–346.
- [141] E.T. Quinto, The invertibility of rotation invariant Radon transforms, *J. Math. Anal. Appl.*, 91(1983), 510-522.
- [142] E. T. Quinto, Tomographic reconstructions from incomplete data—numerical inversion of the exterior Radon transform, *Inverse Problems* 4(1988), 867–876.
- [143] E. T. Quinto, Limited data tomography in non-destructive evaluation, in *Signal Processing Part II: Control Theory And Applications, IMA Volumes In Mathematics And Its Applications* 23, pp. 347–354, Springer-Verlag 1990.
- [144] E. T. Quinto, Computed tomography and Rockets, in “*Mathematical Methods In Tomography,*” *Proceedings, Oberwolfach, 1990*, *Lecture Notes In Mathematics #1497*, pp. 261–268, Springer Verlag, Berlin, New York, 1991.
- [145] E. T. Quinto, Pompeiu transforms on geodesic spheres in real analytic manifolds, *Israel J. Math.* 84(1993), 353–363.

- [146] E. T. Quinto, Singularities of the X-ray transform and limited data tomography in \mathbf{R}^2 and \mathbf{R}^3 , *SIAM J. Math. Anal.* 24(1993), 1215–1225.
- [147] E. T. Quinto, Radon transforms satisfying the Bolker assumption, pp. 245–250 in [52].
- [148] E. T. Quinto, Exterior and Limited Angle Tomography in Non-destructive Evaluation, *Inverse Problems* 14(1998), 339–353.
- [149] E. T. Quinto, private observation, 1998.
- [150] E.T. Quinto, M. Cheney, and P. Kuchment (Editors), *Tomography, Impedance Imaging, and Integral Geometry*, Lectures in Appl. Math., vol. 30, AMS, Providence, RI 1994.
- [151] G.N. Ramachandran and A.V. Lakshminarayanan, Three dimensional reconstruction from radiographs and electron micrographs: applications of convolutions instead of Fourier Transforms, *Proc. Natl. Acad. Sci. USA* 68(1971), 262–277.
- [152] A. G. Ramm and A. I. Zaslavsky, Reconstructing singularities of a function given its Radon transform *Math. Comput. Modelling* 18(1993), 109–138.
- [153] F. Rashid-Farroukhi, K.J.R. Liu, C.A. Berenstein, and D.Walnut, Wavelet-based multiresolution local tomography *IEEE Trans. Image Processing* 6(1997), 1412–1430.
- [154] F. Santosa, Inverse problem holds key to safe, continuous imaging, *SIAM News*, July 1994, 1 and 16-18.
- [155] F. Santosa, M. Vogelius, A backprojection algorithm for electrical impedance imaging, *SIAM J. Appl. Math.*, 50(1990), 216–241.
- [156] L. A. Shepp and S. Srivastava, Computed tomography of PKM and AKM exit cones, *A. T & T. Technical Journal* 65(1986), 78–88.
- [157] I. Shneiberg, Exponential Radon transform, *Doklady Akad. Nauk SSSR*, 320(1991), no.3, 567–571. English translation in *Soviet. Math. Dokl.*
- [158] I. Shneiberg, Exponential Radon transform, pp. 235–246 in [51].
- [159] J. Siciak, Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of \mathbf{C}^n , *Annales Polonici Mathematici*, 22(1969), 145–171.
- [160] K. Smith and F. Keinert, Mathematical foundations of computed tomography, *Applied Optics* 24(1985), 3950–3957.
- [161] K. T. Smith, D. C. Solmon, and S. L. Wagner, Practical and mathematical aspects of the problem of reconstructing objects from radiographs, *Bulletin AMS* 83(1977), no. 6, 1227–1270.
- [162] D. Solmon, Two inverse problems for the exponential Radon transform, in *Inverse Problems in Action*, (P.S. Sabatier, editor), 46–53, Springer Verlag, Berlin 1990.
- [163] D. Solmon, The identification problem for the exponential Radon transform, *Math. Methods in the Applied Sciences*, 18(1995), 687–695.
- [164] E. Somersalo, M. Cheney, D. Isaacson, E. Isaacson, Layer-stripping: a direct numerical method for impedance imaging, *Inverse Problems*, 7(1991), 899–926.
- [165] J. Sylvester, A Convergent Layer-Stripping Algorithm for the Radially Symmetric Impedance Tomography Problem, *Comm. in Part. Diff.Equat.*, 17(1992), 1955–1994.
- [166] J. Sylvester and G. Uhlmann, The Dirichlet to Neumann map and its applications, in “Inverse Problems in Partial Differential Equations”, *SIAM*, 1990, pp. 101–139.
- [167] O.J. Tretiak and C. Metz, The exponential Radon transform, *SIAM J.Appl.Math.* 39(1980), 341–354.
- [168] F. Trèves, *Introduction to Pseudodifferential and Fourier integral operators II*, Plenum Press, New York, 1980.
- [169] H. K. Tuy, An inversion formula for cone-beam reconstruction, *SIAM J. Appl. Math.* 43(1983), 546–552.
- [170] G. Uhlmann, Inverse boundary value problems and applications, *Asterisque* 207(1992), 153–211
- [171] E.T. Vainberg, I.A. Kozak, and M.L. Faingois, Backprojection with second derivative filtration in Roentgen computer tomography, in *Linear and nonlinear problems of computer tomography*, Siberian Branch of Acad. Sci., Novosibirsk Russia, 1985, pp. 16–26.

- [172] L. Zalcman, Offbeat Integral Geometry, *Amer. Math. Monthly* 87(1980), 161–175.