JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **91**, 510–522 (1983)

The Invertibility of Rotation Invariant Radon Transforms

ERIC TODD QUINTO

Department of Mathematics, Tufts University, Medford, Massachusetts 02155

Submitted by P. D. Lax

Let R_μ denote the Radon transform on R^n that integrates a function over hyperplanes in given smooth positive measures μ depending on the hyperplane. We characterize the measures μ for which R_μ is rotation invariant. We prove rotation invariant transforms are all one-to-one and hence invertible on the domain of square integrable functions of compact support, $L_0^2(R^n)$. We prove the hole theorem: if $f \in L_0^2(R^n)$ and $R_\mu f = 0$ for hyperplanes not intersecting a ball centered at the origin, then f is zero outside of that ball. Using the theory of Fourier integral operators, we extend these results to the domain of distributions of compact support on R^n. Our results prove invertibility for a mathematical model of positron emission tomography and imply a hole theorem for the constantly attenuated Radon transform as well as invertibility for other Radon transforms.

1. Introduction

Radon transforms have a rich history and have been studied both for their intrinsic interest [3, 7, 10, 11, 17, 26] as well as for applications to tomography [2, 19–21, 23], partial differential equations [14, 15] and other fields [8]. In Section 2 of this article we characterize the class of rotation invariant Radon transforms on hyperplanes in R^n (Proposition 2.1). This class is a natural generalization of the classical Radon transform which is both rotation and translation invariant (see [17] for another natural generalization). In Section 3 we state and prove the main theorems. Using the theory of Volterra and Abel integral equations we prove Theorem 3.1: All rotation invariant Radon transforms are one-to-one on the domain of square integrable functions of compact support, $L_0^2(R^n)$. The proof implies the hole theorem stated in the abstract and generalizes results in [10, 15, 21]. We also prove that both of these results are true on the domain of compactly supported distributions (Corollary 3.2). The proof rests on the theory of Fourier integral operators. Finally, in Section 4 we discuss how these results pertain to the theoretical aspects of problems in tomography and to inversion of other Radon transforms.

0022-247X/83/020510–13$03.00/0

The support restrictions above are reasonable because the classical Radon transform does have a nontrivial null space for functions not of compact support [18].

2. Definitions and Preliminary Results

Let $\cdot$ denote the standard inner product on R^n, $|\ |$ the induced norm; and let dx be Lebesgue measure on R^n. Let $C^m(R^n)$ be the space of functions continuously differentiable to order $m = 1, 2, \ldots$. Let $C_0(R^n)$ be the space of continuous functions of compact support on R^n. Let S^{n-1} be the unit sphere in R^n and let $d\omega$ and dp denote the standard measures on S^{n-1} and R, respectively. Let ω_n be the volume of S^{n-1} in the measure $d\omega$. Then $L^2(S^{n-1} \times R)$ is the space of square integrable functions in the product measure $d\omega\, dp$. For $(\omega, p) \in S^{n-1} \times R$ we denote the hyperplane perpendicular to ω and containing $p\omega$ by $H(\omega, p) = \{x \in R^n \mid x \cdot \omega = p\}$. Note that $H(\omega, p) = H(-\omega, -p)$. Let dx_H be the measure on $H(\omega, p)$ induced from Lebesgue measure on R^n. We have

Definition 2.1. Let $\mu(x, \omega, p) \in C^\infty(R^n \times S^{n-1} \times R)$ be a strictly positive function such that $\mu(x, \omega, p) = \mu(x, -\omega, -p)$. The *Radon transform* $R_\mu\colon C_0(R^n) \to C_0(S^{n-1} \times R)$ is defined for $f \in C_0(R^n)$ and $(\omega, p) \in S^{n-1} \times R$ by

$$R_\mu f(\omega, p) = \int_{x \in H(\omega, p)} f(x)\, \mu(x, \omega, p)\, dx_H. \tag{2.1}$$

In Section 3 we extend the domain of R_μ to $L_0^2(R^n)$ as well as the class of distributions of compact support.

This definition is a special case of the integral transform defined by Gelfand [7]. The transform R_μ is determined by the values of $\mu(x, \omega, p)$ for $x \in H(\omega, p)$. If f is supported in the ball $\{x \in R^n \mid |x| \leqslant K\}$, then $R_\mu f(\omega, p)$ is supported in the cylinder $\{(\omega, p) \mid |p| \leqslant K\}$. Also $R_\mu f(\omega, p) = R_\mu f(-\omega, -p)$, so $R_\mu f$ can be considered to be a function on hyperplanes.

The transform R_μ is *rotation invariant* if, for each rotation $k \in O(n)$ and each $f \in C_0(R^n)$,

$$R_\mu f(\omega, p) = R_\mu (f \circ k)(k^{-1}\omega, p). \tag{2.2}$$

Proposition 2.2. *Under the assumptions of Definition* 2.1, R_μ *is rotation invariant if and only if there is a function* $U \in C^\infty(R^2)$ *such that* $U(r, p) = U(-r, p) = U(r, -p)$ *and for all* (ω, p) *and all* $x \in H(\omega, p)$, $\mu(x, \omega, p) = U(|x - p\omega|, p)$.

The classical Radon transform on R^n is simply R_μ for $\mu = 1$; this transform is rotation invariant. In [17] the author characterized and inverted "translation invariant" Radon transforms, another generalization of the classical Radon transform. Alexander Hertle solved a related problem in [11] when he showed the classical Radon transform is characterized by its behavior under rotations, translations, and dilations.

Proof. It is straightforward to check that R_μ is rotation invariant for μ related to U as in the proposition.

Assume R_μ is rotation invariant. Let $(\omega, p) \in S^{n-1} \times R$ and let $x_0 \in H(\omega, p)$ and $k \in O(n)$. Let $\delta_{x_0,\omega,p}$ be the Dirac delta function at x_0 on $H(\omega, p)$ (i.e., $\int_{H(\omega,p)} f(x)\, \delta_{x_0,\omega,p}(x)\, dx_H = f(x_0)$ for $f \in C(R^n)$. The distribution $\delta_{x_0,\omega,p}$ can be given as the limit as $\varepsilon \to 0$ of functions $\phi_\varepsilon \in C_0^\infty(R^n)$, where $\{\phi_\varepsilon\}$ is an approximate identity at x_0 *on* $H(\omega, p)$.) Using (2.1) and (2.2), we see

$$\begin{aligned} \mu(x_0, \omega, p) &= R_\mu(\delta_{x_0,\omega,p})(\omega, p) = R_\mu(\delta_{k^{-1}x_0,k^{-1}\omega,p})(k^{-1}\omega, p) \\ &= \mu(k^{-1}x_0, k^{-1}\omega, p). \end{aligned} \tag{2.3}$$

Therefore, for all k that fix ω (and hence $H(\omega, p)$), $\mu(x_0, \omega, p) = \mu(k^{-1}x_0, \omega, p)$ and so μ is a radial function in the first coordinate on $H(\omega, p)$ from the point $p\omega$. Therefore, for some function U and for $x \in H(\omega, p)$, $\mu(x, \omega, p) = U(|x - p\omega|, \omega, p)$. A similar argument, now using all $k \in O(n)$, shows that U is independent of the second argument; i.e., $\mu(x, \omega, p) = U(|x - p\omega|, p)$ for $x \in H(\omega, p)$. Because μ satisfies the hypotheses of Definition 2.1, U satisfies the other requirements of the proposition.

3. The Main Theorems

We now state our main theorems on the invertibility of R_μ. After giving some background information, we prove them. At the end of the section, we discuss how our theorems can be used to get explicit inversion formulas for R_μ.

Recall that a function f is supported in a closed set A if $f(x) = 0$ for $x \notin A$.

THEOREM 3.1. *Let the Radon transform R_μ satisfy Definition* 2.1.

(i) *For each* $M > 0$, $R_\mu : L^2(\{x \in R^n \mid |x| \leqslant M\}) \to L^2(S^{n-1} \times R)$ *is continuous.*

Assume in addition that R_μ is rotation invariant and $f \in L_0^2(R^n)$.

(ii) *If $K > 0$ and $R_\mu f$ is supported in $\{(\omega, p) \mid |p| \leqslant K\}$, then f is supported in $\{x \in R^n \mid |x| \leqslant K\}$.*

(iii) $R_\mu : L_0^2(R^n) \to L^2(S^{n-1} \times R)$ *is one-to-one.*

Part (ii) is the hole theorem; if $f \in L_0^2(R^n)$, then values of f outside of a ball centered at 0 are determined by values of $R_\mu f$ on hyperplanes not intersecting the ball. And (iii) proves that R_μ is invertible on domain $L_0^2(R^n)$.

Let $\mathscr{E}'(R^n)$ (respectively, $\mathscr{E}'(S^{n-1} \times R)$) be the space of distributions of compact support on R^n (respectively, $S^{n-1} \times R$). A distribution u is supported in a closed set A if, for all smooth functions f of compact support that are zero on A, $u(f) = 0$ [9, 12].

We now state a distribution analog of Theorem 3.1.

COROLLARY 3.2. *Let the Radon transform R_μ satisfy Definition* 2.1.

(i) $R_\mu : \mathscr{E}'(R^n) \to \mathscr{E}'(S^{n-1} \times R)$ *is continuous.*

Assume R_μ is rotation invariant and $u \in \mathscr{E}'(R^n)$.

(ii) *If $K > 0$ and $R_\mu u$ is supported in $\{(\omega, p) \mid |p| \leqslant K\}$, then u is supported in $\{x \in R^n \mid |x| \leqslant K\}$.*

(iii) $R_\mu : \mathscr{E}'(R^n) \to \mathscr{E}'(S^{n-1} \times R)$ *is one-to-one.*

Part (ii) of these theorems generalize theorems of Helgason [10], Ludwig [15] and Solmon [21] in which they proved this result for the classical Radon transform and various classes of functions.

Before proving the theorems, we recall some preliminary definitions.

A spherical harmonic on S^{n-1} is the restriction to S^{n-1} of a harmonic polynomial on R^n. Spherical harmonics are dense in $L^2(S^{n-1}, d\omega)$; we let $Y_{lm}(\omega)$, $l \in \mathbb{N}$, $m = 1, 2, ..., N(l)$ be an orthonormal basis, where each Y_{lm} is a homogeneous polynomial of degree l and $N(l)$ is the number of spherical harmonics of degree l in a basis. For $n \geqslant 3$, let $\lambda = (n-2)/2$ and let $C_l^\lambda(t)$ denote the classical Gegenbauer polynomial of degree l. These polynomials are orthogonal on $[-1, 1]$ with weight $(1-t^2)^{\lambda - 1/2}\, dt$. Spherical harmonics on S^1 are simply trigonometric polynomials. The Chebychev polynomial of the first kind of degree l is denoted $T_l(t)$, $l \in \mathbb{N}$ and these polynomials are mutually orthogonal on $[-1, 1]$ with weight $(1-t^2)^{-1/2}\, dt$. For their other properties see [6].

We also need the following results from the theory of integral equations.

THEOREM A. *Let $a < b$ and let $E(s, t) \in C^1([a, b]^2)$, $E(s, s) \neq 0$ for all $s \in [a, b]$. Let $g(t)$ be absolutely continuous for $t \in [a, b]$, $g(a) = 0$, $g' \in L^2([a, b])$. Then the Volterra integral equation*

$$g(t) = \int_a^t E(s, t) f(s)\, ds \tag{3.1}$$

has a unique solution $f \in L^2([a, b])$.

The proof consists in showing that (3.1) is equivalent to a Volterra integral equation of the second kind by taking the first derivative of (3.1). See [24, pp. 10–16] and Lemma 3.3 below.

THEOREM B. *Let $a < b$ and let g and E satisfy the hypotheses of Theorem* A. *Then the generalized Abel integral equation*

$$g(t) = \int_a^t E(s, t)(t - s)^{-1/2} f(s)\, ds \tag{3.2}$$

has a unique solution $f \in L^2([a, b])$.

To prove Theorem B, first multiply both sides of (3.2) by $(u - t)^{-1/2}$ and integrate from a to u. Because the map $f \to \int_a^u (u - t)^{-1/2} f(t)\, dt$ is one-to-one and continuous from $L^2([a, b])$ to $L^2([a, b])$, one can finish the proof as Yosida does in the continuous case [25, pp. 154–157]. See also [22].

Proof of Theorem 3.1. The proof of (i) is left to the reader. It is similar to the proof in [20] for the classical Radon transform.

The proofs of (ii) and (iii) will go as follows: We expand $R_\mu f$ in spherical harmonics and show that its coefficients satisfy certain Volterra or Abel integral equations of the first kind. The nature of these equations imply (ii) and therefore (iii).

Let $f \in L_0^2(R^n)$. Assume $M > K > 0$ such that f is supported in the ball $\{x \in R^n \mid |x| \leqslant M\}$ and $R_\mu f$ is supported in $\{(\omega, p) \mid |p| \leqslant K\}$. Let f_{lm} be the coefficient of Y_{lm} in the spherical harmonic expansion of f; $f_{lm}(r) = \int_{S^{n-1}} f(r\omega)\, Y_{lm}(\omega)\, d\omega$. Then $f_{lm} \in L^2([0, M], r^{n-1}\, dr)$. We now calculate $R_\mu(f_{lm} Y_{lm})$. Let $p > 0$, $\omega \in S^{n-1}$. Let $S^0 = \{\tau \in S^{n-1} \mid \tau \cdot \omega = 0\}$, $S^+ = \{\xi \in S^{n-1} \mid \xi \cdot \omega > 0\}$, and let $d\tau$ be the standard measure on $S^0 \cong S^{n-2}$. For $\tau \in S^0$, $\phi \in [0, \pi/2)$, the map

$$(\tau, \phi) \to (\cos \phi)\, \omega + (\sin \phi)\, \tau = \xi \tag{3.3}$$

gives coordinates on S^+ in which the standard measure $d\xi$ can be written $d\xi = (\sin \phi)^{n-2}\, d\phi\, d\tau$. Furthermore, $\cos \phi = \xi \cdot \omega$. In a similar manner

$$(\tau, \phi) \to p \sec \phi((\cos \phi)\, \omega + (\sin \phi)\, \tau) = x \tag{3.4}$$

gives coordinates on $H(\omega, p)$ in which $dx_H = (p^{n-1}/(\cos \phi)^n)(\sin \phi)^{n-2}\, d\phi\, d\tau$. Therefore, under the map $\chi\colon S^+ \to H(\omega, p)$, $\chi(\xi) = (p/(\xi \cdot \omega))\, \xi$, the pull back to S^+ of dx_H is $\chi^*\, dx_H = p^{n-1}(\xi \cdot \omega)^{-n}\, d\xi$. This proves

$$\begin{aligned} R_\mu(f_{lm} Y_{lm})(\omega, p) = \int_{\xi \in S^+} & f_{lm}(p/(\xi \cdot \omega))\, Y_{lm}(\xi) \\ & \times U(p(1 - (\xi \cdot \omega)^2)^{1/2}/(\xi \cdot \omega), p)(p^{n-1}/(\xi \cdot \omega)^n)\, d\xi, \end{aligned} \tag{3.5}$$

where U is as in Proposition 2.2. Using the Funk–Hecke Theorem [6] for $n \geqslant 3$, $\lambda = (n-2)/2$, we get

$$R_\mu(f_{lm} Y_{lm})(\omega, p) = \frac{\omega_{n-1}}{C_l^\lambda(1)} Y_{lm}(\omega) \int_0^1 C_l^\lambda(r) f_{lm}(p/r) \times U(p(1-r^2)^{1/2}/r, p)(p^{n-1}/r^n)(1-r^2)^{\lambda-(1/2)}\, dr. \quad (3.6)$$

Therefore the integral in (3.6) is the coefficient of Y_{lm} in the spherical harmonic series for $R_\mu f$. We call that coefficient $(R_\mu f)_{lm}$. Changing variables $s = r/p$ in the integral in (3.6) and letting $t = 1/p$, we see

$$(R_\mu f)_{lm}(1/t) = \frac{\omega_{n-1}}{C_l^\lambda(1)} \int_a^t C_l^\lambda\left(\frac{s}{t}\right) f_{lm}\left(\frac{1}{s}\right) \times U((1-(s/t)^2)^{1/2}/s,\, 1/t)\, s^{-n}(1-(s/t)^2)^{\lambda-1/2}\, ds, \quad (3.7)$$

where $0 < a < 1/M$. We can integrate from a to t because $f_{lm}(t) = 0$ for $t > M$. Following the above arguments for $n = 2$, we conclude

$$(R_\mu f)_{lm}(1/t) = 2 \int_a^t T_l(s/t) f_{lm}(1/s) \times U((1-(s/t)^2)^{1/2}/s,\, 1/t)\, s^{-2}(1-(s/t)^2)^{-1/2}\, ds, \quad (3.8)$$

where $0 < a < 1/M$.

We now prove (ii). Let $b = 1/K$. By hypothesis, $f_{lm}(1/s) \in L^2([a, b])$ and $(R_\mu f)_{lm}(1/t) = 0$ for $t < b = 1/K$. Recall that $U(r, p)$ is a smooth positive function satisfying $U(r, p) = U(-r, p)$. Therefore, for $n = 2, 3, \ldots$,

$$0 = (R_\mu f)_{lm}(1/t) = \int_a^t W(s, t) f_{lm}(1/s)(t-s)^{(n-3)/2}\, ds \quad (3.9)$$

for some function $W \in C^\infty([a, b]^2)$, $W(s, s) \neq 0$ for all $s \in [a, b]$.

For $n = 2$, we can apply Theorem B to conclude that $f_{lm}(1/s) = 0$ for $s \in [a, b]$ and so $f_{lm}(r) = 0$ for $r > K = 1/b$. This proves (ii) for $n = 2$. We use Theorem A to prove (ii) for $n = 3$. To prove (ii) for $n > 3$ we use

LEMMA 3.3. *If $K(s, t) \in C^1([a, b]^2)$, $f \in L^2([a, b])$, then*

$$\int_a^t K(s, t) f(s)\, ds \quad (3.10)$$

is absolutely continuous on $[a, b]$ and has first derivative

$$K(t, t) f(t) + \int_a^t \frac{\partial K}{\partial t}(s, t) f(s)\, ds. \quad (3.11)$$

This is proven by showing that the integral from a to x of (3.11) is (3.10). This is clearly true for f continuous and is true for $f \in L^2$ by continuity.

If n is even, $n > 2$, then we use Lemma 3.3 to take $(n-2)/2$ derivatives of (3.9) with respect to t. This gives an integral equation of the form (3.2) [25, pp. 153–154] and Theorem B proves that $f_{lm}(r) = 0$ for $r > K = 1/b$. For odd $n > 3$ the proof is similar. Equation (3.9) is reduced to (3.1) by taking $(n-3)/2$ derivatives. Therefore (ii) is true for all n. This implies (iii) and finishes the proof of the theorem.

Proof of Corollary 3.2. It is well known that R_μ is an elliptic Fourier integral operator [9]. In [17] we considered R_μ as an operator from $C_0^\infty(R^n)$ to $C_0^\infty(Y)$, where $Y = S^{n-1} \times R/\sim$, where $\sim$ is the equivalence relation $(\omega, p) \sim (-\omega, -p)$. We denote the equivalence class of (ω, p) by $[\omega, p]$. The local canonical graph $\Gamma \subset T^*(R^n \times Y)$ associated to R_μ is a set that lies above

$$Z = \{(x, [\omega, p]) \in R^n \times Y \mid x \cdot \omega = p\} \tag{3.12}$$

[17, p. 329]. Because Γ is a conic local canonical graph and R_μ preserves compact support, R_μ is continuous from $\mathscr{E}'(R^n) \to \mathscr{E}'(S^{n-1} \times R)$ [12, p. 130]. This proves (i).

Recall that the singular support, sing supp, of a distribution is the set of points near which the distribution is not given by a smooth function (see [12]).

We now prove (ii). Let R_μ be a rotation invariant Radon transform. Let $u \in \mathscr{E}'(R^n)$; choose $M > 0$ such that the support of u lies in the ball $A = \{x \in R^n \| x| < M\}$. Assume

$$R_\mu u \text{ is supported in } \qquad B = \{[\omega, p] \in Y \mid |p| \leqslant K\}, \qquad K < M. \tag{3.13}$$

Then R_μ can be changed off A to become a properly supported [12] operator that agrees with R_μ on $\mathscr{E}'(A)$. By the conclusion of [9, Proposition 6.5], which is valid for R_μ, there is a Fourier integral operator S such that the pseudodifferential operator SR_μ is elliptic and hence preserves singular support on $\mathscr{E}'(A)$. The local canonical graph associated to S is Γ^t, where Γ^t is Γ with T^*R^n and T^*Y coordinates reversed. Using [12, (2.5.9)], the wave front set of u can be calculated in terms of Γ^t and the wave front set of $R_\mu u$. From that calculation and the fact that sing supp u is the projection to R^n of the wave front set of u [12, Theorem 2.5.3], we see

$$\text{sing supp } u \subset \{x \in R^n \mid \text{for some } [\omega, p] \in \text{sing supp } R_\mu u, \ (x, [\omega, p]) \in Z\}. \tag{3.14}$$

By hypothesis (3.13), sing supp $R_\mu u \subset B$. Using expression (3.14) for sing supp u together with (3.12), we see sing supp $u \subset \{x \in R^n \mid |x| \leqslant K\}$.

Therefore $u(x)$ is smooth for $|x| > K$. Since u has compact support, we can use Theorem 3.1(ii) to conclude u must be zero for $|x| > K$. This proves (ii).

A variation on this argument shows that the null space of R_μ on domain $\mathscr{E}'(R^n)$ consists entirely of smooth functions of compact support. By Theorem 3.1(iii) this null space is $\{0\}$.

Remarks on Inversion Formulas for R_μ

One method of getting an inversion formula for R_μ is to invert each integral equation (3.7) explicitly. For the classical Radon transform this has been done by Cormack [2] on R^2 and by S. R. Deans [5] on R^n for $n > 2$. For each l, Deans found an integral operator involving C_l^λ that, when composed with the operator in (3.7), produces an easily invertible operator (see [5, Eq. (17)]). Cormack's operator involves T_l.

Perhaps one can invert some R_μ using orthogonal expansions (see [16]). Alternatively, to invert R_μ, one can take derivatives of (3.9) as in the proof of Theorem 3.1 to get a Volterra integral equation and then use its Neumann series [14, 25] to invert it. For even n one first needs to convert the resulting Abel equation (3.2) to a Volterra equation [25, pp. 154–157]. Even when the Neumann series cannot be summed, at least it might provide information about the asymptotic behavior of the solution.

Finally, since R_μ is an elliptic Fourier integral operator, one can try to invert it using the theory of pseudodifferential and Fourier integral operators as the author did in [17] for another class of Radon transforms.

4. Applications

The first application is to a theoretical problem related to positron emission tomography (PET).

Example 4.1 *PET Scanning.* Scintillation detectors are placed in a circle around a positron-emitting object. If x is a point in the plane of the detectors, we assume emissions at x occur at the same time in opposite directions [1] and there is an equal probability for emissions to travel along any line through x. Let $f(x)$ be the function giving the concentration of emitters at x. If detectors at A radians and B radians on the circle detect an emission at almost the same time, it is assumed that positrons (and then gamma rays) were emitted somewhere along the line segment H between A and B. The number of emissions detected almost simultaneously at A and B should be proportional to the integral of f over H in Lebesgue measure dx_H. Because the detectors are not point detectors, however, the probability that an emission on H is detected at both A and B is related to the position of the emitter along H. Assume the detector at A runs from $A - d$ radians to $A + d$

radians ($d > 0$) along the circle enclosing the object and that the detector at B runs from $B - d$ to $B + d$ radians along the circle. If an emission occurs at $x \in H$, it will be recorded at both A and B only if the line the gamma rays travel along intersects both arcs $(A - d, A + d)$ and $(B - d, B + d)$. Furthermore, each detector itself is more sensitive at the center than at the edges [1, pp. 159–160]. Therefore an emission at the midpoint of H is more likely to be detected at A and B than one near the ends of H. This probability will modify the integral above by a function μ symmetric about the midpoint of H. The function μ is rotation invariant about the center of the circle of detectors as a function of the line H. Hence the number of emissions detected almost simultaneously at A and B is proportional to the integral of f over H in the measure $\mu\, dx_H$. Therefore this example can be modeled as the discretization of the rotation invariant Radon transform R_μ on R^2. We point out that scatter and random coincidences are problems inherent in PET [1] that are not addressed in this model. It is possible that they can only be resolved practically.

By Theorem 3.1 this transform R_μ is invertible. The sort of inversion formulas gotten from the integral equations of Theorem 3.1 might not be suitable for practical computer reconstructions; they can become unstable for large l as well as for reconstructions near the origin ([2, Eq. (18); 5, Eq. (17)] are inversion formulas for the classical Radon transform gotten from equations equivalent to (3.7). The kernel of the inverse operators do blow up for large l and for x close to the origin).

Because R_μ is invertible and satisfies the hole theorem (Theorem 3.1(ii)), however, it is reasonable to try to find other inversion algorithms for this example and, perhaps, even algorithms that only use integrals over lines away from the center to reconstruct the concentration of emitters away from the center (see [19, p. 425]) (after having calculated μ!).

Our next example was discussed at the 1981 conference on Radon transforms and their applications at Tufts University.

EXAMPLE 4.2. *The Constantly Attenuated Radon Transform on R^2.* The attenuated Radon tranform is a model for single photon emission tomography with constant attenuation [23]. Let $\theta \in [0, 2\pi]$ and let $\bar{\theta} = (\cos\theta, \sin\theta)$, $\theta^\perp = (-\sin\theta, \cos\theta)$. Let $M, p \in R, f \in L_0^2(R^2)$. The constantly attenuated Radon transform evaluated on f is

$$T_M f(\bar{\theta}, p) = \int_{-\infty}^{\infty} f(p\bar{\theta} + s\theta^\perp)\, e^{Ms}\, ds. \tag{4.1}$$

Values of $T_M f$ correspond to the radioactivity measured directionally outside the body. The transform T_M is not rotation invariant but

$$\tfrac{1}{2}(T_M f(\bar{\theta}, p) + T_M f(-\bar{\theta}, -p)) = R_\mu f(\bar{\theta}, p) \tag{4.2}$$

where

$$\mu(x, \bar{\theta}, p) = \cosh |x - p\bar{\theta}|.$$

Because this R_μ is rotation invariant, we find the known result that T_M is one-to-one on $L_0^2(R^2)$ [23]. We learn, however, the new result that T_M also satisfies the hole theorem (Theorem 3.1(ii)).

It is interesting to note that averaging like (4.2) has already been done, independently, for practical reasons [13]. Apparently it decreases the effects of attenuation when using the standard Fourier inversion algorithm on single photon emmision tomography [13]. Exact inversion methods for T_M are given in [23] and more recently by A. Markoe [27].

The trick of Eq. (4.2) can be used to prove invertibility for any transform

$$T_N f(\bar{\theta}, p) = \int_{-\infty}^{\infty} f(p\bar{\theta} + s\theta^\perp) N(s, p) \, ds$$

for which $N(s, p) = N(s, -p)$, $N \in C^\infty(R^2)$, $N > 0$.

EXAMPLE 4.3 *More general Radon transforms.* Radon transforms have been defined in much more general settings than we have discussed [4, 7, 8, 10, 26]. We first give an extension to transforms integrating over other manifolds than hyperplanes in R^n. Let $(\omega, p) \in S^{n-1} \times R$ and let $L(\omega, p)$ be a specified submanifold of R^n (perhaps a hyperboloid [3] or sphere passing through the origin [4, 18]). If a measure is given on each $L(\omega, p)$, one can define a Radon transform S that takes a function $f \in C_0(R^n)$ to its integrals over the $L(\omega, p)$ in their given measures. A natural question to ask is whether S is invertible. Using the techniques above, we can answer this question in some cases.

The crux of the proof of Theorem 3.1 is to write the integral of $R_\mu f$ over a hyperplane as an integral over a sphere (3.5) and, if the measure μ is "nice" (rotation invariant), use the Funk–Hecke theorem to simplify the integral over the sphere. This results in invertible integral equations (3.9) involving the spherical harmonic coefficients of f.

In many cases this procedure can be used for the transform S above. If

each $L(\omega, p)$ can be mapped to a cap of the sphere $\{\xi \in S^{n-1} \mid \xi \cdot \omega > a\}$ for some a, perhaps depending on p, (4.3)

and

there is a positive function $m \in C^\infty(R^2)$ such that the smooth measures on all $L(\omega, p)$ project onto measures $m(\omega \cdot \xi, p) \, d\xi$ on the caps, (4.4)

then the Funk–Hecke theorem can be used, as above, to get integral equations involving the spherical harmonic coefficients of the function f. If the equations are invertible (e.g., of the form (3.9)), then the transform is invertible.

These ideas were behind the result of Cormack and the author in [4] that produced an inversion formula for the Radon transform on spheres through the origin in R^n. Cormack has used this idea to get nice inversion formulas for integrals over various parameterized families of curves in the plane [3] as well as in R^n [unpublished work]. Moreover, this procedure can certainly be done on any of the above manifolds with any smooth positive measures that satisfy (4.4). Such measures are rotation invariant.

The Radon transform that integrates over "V's" in R^2 is another example that can be inverted in the manner outlined above. Let $\alpha \in (0, \pi/2)$ and $(\theta, p) \in [0, 2\pi] \times (0, \infty)$; then $L(\bar{\theta}, p) = \{x \in R^2 \mid x = p\bar{\theta} + t(\overline{\theta + \alpha})$ or $x = p\bar{\theta} + t(\overline{\theta - \alpha})$ for some $t \geqslant 0\}$ is a "V" with vertex $p\bar{\theta}$ and angle 2α (notation as in Example 4.2). For $f \in C_0(R^2)$, let $Sf(\bar{\theta}, p)$ be the integral of f over $L(\bar{\theta}, p)$ in its standard measure, arc length. By performing (4.3), it is a simple exercise to calculate the spherical harmonic coefficients (Fourier coefficients) of Sf in terms of those of f. Then a change of variable shows that the coefficients of f satisfy invertible integral equations of the form (3.1). The nature of the equations shows the hole theorem: The integrals of f over all V's, $L(\bar{\theta}, p)$, not intersecting a ball centered at the origin in R^2 determine f outside of that ball.

Our final example involves the Radon transform on real projective space, RP^n [10]. We project it onto a rotation invariant Radon transform on R^n. Consider RP^n as the set of lines through the origin in R^{n+1}. For $\xi \in S^n$, the projective hyperplane $H(\xi)$ is the set of lines through the origin in R^{n+1} perpendicular to ξ. Let $e_{n+1} = (0, \ldots, 0, 1) \in R^{n+1}$ and let ψ be affine projection from $RP^n - H(e_{n+1})$ to $\{(x_1, \ldots, x_{n+1}) \in R^{n+1} \mid x_{n+1} = 1\} \cong R^n$. If $\xi \neq e_{n+1}$, then $\psi(H(\xi))$ is an $(n-1)$-dimensional hyperplane in R^n and the standard measure on $H(\xi)$ [10] projects onto a rotation invariant measure on this hyperplane (the measure is closely related to dx_H). Therefore if $f \in C(RP^n)$, then its Radon transform evaluated at $H(\xi)$ is the same as a specific rotation invariant Radon transform on R^n of $f \circ \psi^{-1}$ evaluated at the hyperplane $\psi(H(\xi))$. Taking $\eta = e_{n+1}$ and using Theorem 3.1 gives the following hole theorem:

PROPOSITION 4.1. *Let $\eta \in S^n$, $d > 0$. If $f \in C(RP^n)$ is zero on some neighborhood of $H(\eta)$ and its Radon transform is zero on all projective hyperplanes $H(\xi)$ for which $|\eta - \xi| < d$, then f is zero on $\bigcup_{|\xi - \eta| < d} H(\xi)$.*

It is well known that this Radon transform on RP^n is invertible [10].

ACKNOWLEDGMENTS

The author would like to thank Allan Cormack for suggesting this problem and pointing out Example 4.1, one motivation of this paper. The author also thanks Carlos Berenstein, Rod Deans, and Lawrence Shepp for their information as well as Charles Hamaker, James Peters, and Allan Cormack for helpful suggestions about the article.

REFERENCES

1. R. A. BROOKS, V. J. SANK, W. F. FRIAUF, S. B. LEIGHTON, H. E. CASCIO, AND G. DICHIRO, Design considerations for positron emission tomography, *IEEE Trans. Biomed. Engrg.* **28** (1981), 158–177.
2. A. M. CORMACK, Representation of a function by its line integrals, with some radiological applications, *J. Appl. Phys.* **34** (1963), 2722–2727.
3. A. M. CORMACK, The Radon transform on a family of curves in the plane, *Proc. Amer. Math. Soc.* **86** (1982), 293–298.
4. A. M. CORMACK AND E. T. QUINTO, A Radon transform on spheres through the origin in R^n, *Trans. Amer. Math. Soc.* **260** (1980), 575–581.
5. S. R. DEANS, A unified Radon inversion formula, *J. Math. Phys.* **19** (1978), 2346–2349.
6. A. ERDELYI, W. MAGNUS, R. OBERHETTINGER, AND F. TRICOMI, "Higher Transcendental Functions," Vol. II, McGraw–Hill, New York, 1953.
7. I. M. GELFAND, M. I. GRAEV, AND Z. YA. SHAPIRO, Differential forms and integral geometry, *Funkcional. Anal. i Priložen*, **3** (1969), 24–40; English translation in *Functional Anal. Appl.* **3** (1969), 101–114.
8. V. GUILLEMIN, The Radon transform on Zoll surfaces, *Advan. Math.* **22** (1976), 85–119.
9. V. GUILLEMIN AND S. STERNBERG, "Geometric Asymptotics," Math. Surveys No. 14, American Mathematical Society, Providence, R. I., 1977.
10. S. HELGASON, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces, and Grassman manifolds, *Acta Math.* **113** (1965), 153–180.
11. A. HERTLE, A Characterization of Fourier and Radon Transforms on Euclidean Space, *Trans. Amer. Math. Soc.* **273** (1982), 595–607.
12. L. HÖRMANDER, Fourier integral operators, I, *Acta Math.* **127** (1971), 79–183.
13. D. B. KAY AND J. W. KEYES, JR., First-order corrections for absorption and resolution compensation in radionuclide Fourier tomography, *J. Nuclear Med.* **16** (1975), 540–541.
14. P. D. LAX AND R. S. PHILLIPS, The Paley–Wiener theorem for the Radon transform, *Comm. Pure Appl. Math.* **23** (1970), 409–424.
15. D. LUDWIG, The Radon transform on Euclidean space, *Comm. Pure Appl. Math.* **19** (1966), 49–81.
16. R. M. PERRY, On reconstructing a function on the exterior of a disc from its Radon transform, *J. Math. Anal. Appl.* **59** (1977), 324–341.
17. E. T. QUINTO, The dependence of the generalized Radon transform on defining measures, *Trans. Amer. Math. Soc.* **257** (1980), 331–346.
18. E. T. QUINTO, Null spaces and ranges for the classical and spherical Radon transforms, *J. Math. Anal. Appl.*, to appear.
19. L. A. SHEPP AND J. B. KRUSKAL, Computerized tomography: the new medical X-ray technology, *Amer. Math. Monthly* **85** (1978), 420–439.
20. K. T. SMITH, D. C. SOLMON, AND S. L. WAGNER, Practical and mathematical aspects of the problem of reconstructing objects from radiographs, *Bull. Amer. Math. Soc.* **83** (1977), 1227–1270.
21. D. C. SOLMON, The X-ray transform, *J. Math. Anal. Appl.* **56** (1976), 61–83.

22. L. TONELLI, Su un problema di Abel, *Math. Ann.* **99** (1929), 183–199.
23. O. TRETIAK AND C. METZ, The exponential Radon transform, *SIAM J. Appl. Math.* **39** (1980), 341–354.
24. F. G. TRICOMI, "Integral Equations," Interscience, New York, 1957.
25. K. YOSIDA, "Lectures on Differential and Integral Equations," Interscience, New York, 1960.
26. L. ZALCMAN, Offbeat integral geometry, *Amer. Math. Monthly* **87** (1980), 161–175.
27. A. MARKOE, Fourier inversion of the attenuated X-ray transform, *SIAM J. Math. Anal.*, to appear.

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **94**, 602–603 (1983)

Erratum

Volume **91**, Number 2 (1983), in the article "The Invertibility of Rotation Invariant Radon Transforms," by Eric Todd Quinto, pp. 510–522.

The author has discovered an error in the proof of Corollary 3.2(ii) of [2] (in general, the conclusion below (3.14) does not follow from (3.14)). This note presents a correction that proves a stronger theorem.

Let R_μ be a Radon transform that integrates over hyperplanes in R^n with measure $\mu(x, \omega, p)\, dx_H$ on the hyperplane $H(\omega, p)$ as in Definition 2.1 [2]. If R_μ is rotation invariant, then $\mu(x, \omega, p) = U(|x - p\omega|, p)$ for a function U as in Proposition 2.2 [2]. We have:

THEOREM. *Let R_μ be a rotation invariant Radon transform satisfying $\mu(p\omega, \omega, p) \neq 0$ for all (ω, p). Let $u \in \mathscr{E}'(R^n)$ and $K \geqslant 0$. Assume $R_\mu u$ is supported in $\{(\omega, p) \mid |p| \leqslant K\}$. Then u is supported in $\{x \in R^n \mid |x| \leqslant K\}$.*

This generalizes Theorem 3.1(ii) as well as Corollary 3.2(ii) because of the weaker assumptions here; μ is not required to be positive.

Proof. Let $K > 0$. Let $g \in C^\infty(S^{n-1} \times R)$, then $R_\mu^* g(x) = \int_{S^{n-1}} g(\omega, x \cdot \omega) \mu(x, \omega, x \cdot \omega)\, d\omega$ and $R_\mu^*: C^\infty(S^{n-1} \times R) \to C^\infty(R^n)$ is continuous [1]. Let $g(\omega, p) = g_l(p)\, Y_l(\omega)$, where Y_l is a homogeneous spherical harmonic of degree l and g_l is even if l is even, odd of l is odd. Then, by the nature of $\mu = U$, the Funk–Hecke theorem proves

$$R_\mu^* g(x) = \frac{2\omega_{n-1}}{C_l^\lambda(1)} Y_l(x') \int_0^r r^{-1} g_l(v)\, C_l^\lambda(v/r) \times U(\sqrt{r^2 - v^2}, v)(1 - (v/r)^2)^{(n-3)/2}\, dv, \tag{1}$$

where $x' = x/|x|$, $r = |x|$, and the other notation is as in [2].

This is the key to the proof of:

LEMMA. *Let R_μ be a rotation invariant Radon transform and let $K > 0$. Let $l \in N \cup \{0\}$ and let $f_l(r) \in C^\infty([0, \infty))$ be equal to 0 for $|r| \leqslant K$, then there is a unique $g_l \in C^\infty(R)$ equal to 0 for $|p| \leqslant K$ and even for l even, odd for l odd such that for any homogeneous spherical harmonic of degree l, Y_l,*

$$R_\mu^*(g_l p)\, Y_l(\omega))(x) = f_l(|x|)\, Y_l(x'). \tag{2}$$

0022-247X/83 $3.00

Proof. Because $f_l = 0$ on $[0, K]$ the lower limit of integration in (1) can be taken to be $K/2$. One solves the integral equation in (1) for g_l in terms of f_l as outlined in [2], by taking derivatives and using Theorem A or B. Because the kernel of the resulting integral equation is smooth and f_l is, so is the solution (for $(n-3)/2$ odd, the forcing term [3, 41.6] in the resulting Abel equation [3, 41.4′] is smooth because $f_l = 0$ near $K/2$). By uniqueness, the solution is zero for $|p| \leqslant K$. This proves the lemma.

Let $u \in \mathscr{E}'(R^n)$. Let $f_l(r) \in C^\infty([0, \infty))$ be supported in (K, ∞) and let Y_l be a spherical harmonic. Let $g_l(p)$ satisfy (2). Therefore $\langle u, f_l Y_l \rangle = \langle u, R_\mu^*(g_l(p)\, Y_l(\omega)) \rangle = \langle R_\mu u, g_l Y_l \rangle = 0$ as g_l is supported in (K, ∞). As sums of $f_l Y_l$ are dense in the set of C^∞ functions supported in $\{x \mid |x| > K\}$, u is supported in $\{x \mid |x| \leqslant K\}$. The case $K = 0$ follows immediately.

References

1. E. T. Quinto, The dependence of the generalized Radon transform on defining measures, *Trans. Amer. Math. Soc.* **257** (1980), 331–346.
2. E. T. Quinto, The invertibility of rotation invariant Radon transforms, *J. Math. Anal. Appl.* **91** (1983), 510–522.
3. K. Yosida, "Lectures on Differential and Integral Equations," Interscience, New York, 1960.