Abstract. Here we present a novel microlocal analysis of a new toric section transform which describes a two dimensional image reconstruction problem in Compton scattering tomography and airport baggage screening. By an analysis of two separate limited data problems for the circle transform and using microlocal analysis, we show that the canonical relation of the toric section transform is 2–1. This implies that there are image artefacts in the reconstruction. We provide explicit expressions for the expected artefacts and demonstrate these by simulations. In addition, injectivity proofs are given for smooth functions which are compactly supported on an annulus and we present simulated reconstructions using a discrete approach with varying levels of added pseudo random noise.

1. Introduction

We consider the acquisition geometry displayed in figure 1, which illustrates an idealised source–detector geometry in airport baggage screening representing the Real Time Tomography (RRT) geometry [18]. The inner circle (of smaller radius) represents a ring of fixed energy sensitive detectors and the outer circle a ring of fixed, switched X-ray sources, which we will assume for the purposes of this paper can be simulated to be monochromatic (e.g. by varying the X-ray tube voltage and taking finite differences in energy or by source filtering [7, 8]). It is noted that the RTT geometry is three dimensional [18], but we assume a two dimensional scattering geometry as done in [19].

Compton scattering describes the inelastic scattering process of a photon with charged particles (usually electrons). The energy loss is given by the equation

\[ E' = \frac{E}{1 + (E/E_0)(1 - \cos \omega)}, \]

where \( E' \) is the scattered energy, \( E \) is the initial energy, \( \omega \) is the scattering angle and \( E_0 \) denotes the electron rest energy. If the source is monochromatic (\( E \) is fixed) and we can measure the scattered energy \( E' \), then the scattering angle \( \omega \) of the interaction is determined by equation (1.1). This implies that the locus of Compton scatterers in the plane is a toric section \( T = C_1 \cup C_2 \) (the union of two intersecting circles which have the same radius). This is the idea behind two dimensional Compton scattering tomography [19, 14, 16, 15]. It can be shown [19], that if the attenuation effects are ignored, the Compton scattered intensity can be modelled as the integral of the charge density (electron density) over toric sections. We make the same assumption in this paper, and model the scattered intensity measured by the RTT system as integrals of the charge density \( f \) (represented by a real valued function) over a set of toric sections in the plane. In three dimensions, the surface of scatterers is described by the surface of revolution of a toric section about its central axis, namely a spindle torus. In [21, 20] the inversion and microlocal aspects of a spindle torus integral transform are considered.
The set of toric sections whose tips (the points of intersection of $C_1$ and $C_2$) lie on two circles (as in figure 1) is three dimensional. Indeed we can vary a source and detector coordinate on $S^1 \times S^1$ and the radius of the circles $r$. In this paper we consider the two dimensional subset of toric sections whose central axis (the line through the points of intersection of $C_1$ and $C_2$) intersects the origin. This can be parametrised by a rotation about the origin ($\theta \in S^1$) and the radius $r \geq 2$, as we shall see later in section 3.

In [19] the RTT geometry is considered and the scattered intensity is approximated as a set of integrals over discs whose boundaries intersect a given source point, and inversion techniques and stability estimates are derived through an equivalence with the Radon transform. Here we present a novel toric section transform (which describes the scattered intensity exactly) and analyse its stability from a microlocal standpoint. So far the results of Natterer [13] have been used to derive Sobolev space estimates for the disc transform presented in [19], but the microlocal aspects of the RTT geometry in Compton tomography are lacking in the literature. We aim to address this here. We explain the expected artefacts in a reconstruction from toric section integral data through an analysis of the canonical relation of a toric section transform, and injectivity results are provided for smooth functions compactly supported on an annulus. The expected artefacts are shown through simulation and are as predicted by the theory. We also give simulated reconstructions of two test phantoms (one simple and one complex) with varying levels of added pseudo random noise. In [21] it is suggested to use a Total Variation (TV) regularisation technique to combat the artefacts in a three dimensional

![Figure 1: A toric section $T = C_1 \cup C_2$ with axis of rotation $\theta = (0, 1)$, tube centre offset $s = \sqrt{r^2 - 4}$ and tube radius $r$. The coordinates $c_1$ and $c_2$ denote the centers of the circles $C_1$ and $C_2$ respectively.](image)
Compton tomography problem. Here we show that we can combat the artefacts present in the reconstruction effectively in two dimensions using a discrete approach with a heuristic iterative solver and a TV regulariser.

In section 2 we recall some definitions and results on Fourier Integral Operators (FIO’s) and microlocal analysis before introducing a new toric section transform in section 3, which describes the Compton scattered intensity collected by the acquisition geometry in figure 1. Later in section 3.1 we provide a novel microlocal analysis of the toric section transform when considered as an FIO. Through an analysis of the canonical relations of two circle transforms separately (whose sum is equivalent to the toric section transform), we show that the canonical relation of the toric section transform is 2–1 and provide explicit expressions for the artefacts expected in a reconstruction from toric section integral data.

In section 3.2 we prove the injectivity of the toric section transform on the set of smooth functions compactly supported on an annulus. This uses a similar parameterisation of circular arcs to Nguyen and Truong in [14] and proves the injectivity by a decomposition into the Fourier series components and using the ideas of Cormack [1].

In section 4, we present a practical reconstruction algorithm for the recovery of two-dimensional densities from toric section integral data and provide simulated reconstructions of two test phantoms (one simple and one complex) with varying level of added pseudo random noise. Here we use a discrete approach. That is we discretize the toric section integral operator (stored as a sparse matrix) on a pixel grid (assuming a piecewise constant density) and use an iterative technique (e.g. a conjugate gradient method) to solve the sparse set of linear equations described by the discretized operator with regularisation (e.g. Tikhonov or total variation). We demonstrate the artefacts in the reconstruction by an application of the discretized normal operator \( A^T A \), where \( A \) is the discrete form of the toric section transform) to a delta function, and show that the artefacts are exactly as predicted by the theory presented in section 3.1 by a side by side comparison. We further show that we can effectively combat the reconstruction artefacts by applying a total variation penalty and using the heuristic iterative technique of [3] (see also [6]).

2. Microlocal definitions

We next recall some definitions.

**Definition 2.1** ([10, Definition 7.1.1]). For a function \( f \) in the Schwartz space \( S(\mathbb{R}^n) \) we define the Fourier transform and its inverse as

\[
\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,
\]

\[
\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.
\]

We use the standard multi-index notation; if \( f \) is a function on \( \mathbb{R}^n \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0, 1, 2, \ldots\}^n \) is a multi-index, then \( \partial^\alpha f = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n} f \).

We identify cotangent spaces on Euclidean spaces with the underlying Euclidean spaces so if \( X \) is an open subset of \( \mathbb{R}^n \) and \( (x, \xi) \in X \times \mathbb{R}^N \) then \( T^*_x (X \times \mathbb{R}^N) \) is identified with \( \mathbb{R}^n \times \mathbb{R}^N \). Under this identification, if \( \phi = \phi(x, \xi) \) for \( (x, \xi) \in X \times \mathbb{R}^N \) then

\[
d_x \phi = \left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n} \right),
\]

\[
d_\xi \phi = \left( \frac{\partial \phi}{\partial \xi_1}, \frac{\partial \phi}{\partial \xi_2}, \ldots, \frac{\partial \phi}{\partial \xi_N} \right)
\]

and \( d\phi(x, \xi) = (d_x \phi(x, \xi), d_\xi \phi(x, \xi)) \in \mathbb{R}^n \times \mathbb{R}^N \).
Definition 2.2 ([10, Definition 7.8.1]). Let $X$ be an open subset of $\mathbb{R}^n$ $m \in \mathbb{R}$. Then we define $S^m(X \times \mathbb{R}^N)$ to be the set of $a \in C^\infty(X \times \mathbb{R}^N)$ such that for every compact set $K \subset X$ and all multi–indices $\alpha, \beta$ the bound
\[
|\partial^\beta_x \partial^\alpha_\xi a(x, \xi)| \leq C_{\alpha, \beta, K}(1 + |\xi|)^{|\alpha|-m}, \quad x \in K, \ \xi \in \mathbb{R}^n,
\]
holds for some constant $C_K$. The elements of $S^m$ are called symbols of order $m$.

Note that these symbols are sometimes denoted $S^m_{1,0}$

**Definition 2.3** ([11, Definition 21.2.15]). A function $\phi = \phi(x, \xi) \in C^\infty(X \times \mathbb{R}^N \setminus 0)$ is a phase function if $\phi(x, \lambda \xi) = \lambda \phi(x, \xi), \forall \lambda > 0$ and $d\phi$ is nowhere zero. A phase function is clean if the critical set $\Sigma_\phi = \{(x, \xi) : d\xi \phi(x, \xi) = 0\}$ is a smooth manifold with tangent space defined by $d(d\xi \phi) = 0$.

By the implicit function theorem the requirement for a phase function to be clean is satisfied if $d(d\xi \phi)$ has constant rank.

**Definition 2.4** ([11, Definition 21.2.15] and [12, Section 25.2]). Let $X \subset \mathbb{R}^{n_1}$, $Y \subset \mathbb{R}^{n_2}$ be open sets. Let $\phi \in C^\infty(X \times Y \times (\mathbb{R}^N \setminus 0))$ be a clean phase function. Then, the critical set of $\phi$ is
\[
\Sigma_\phi = \{(x, y, \xi) \in X \times Y \times \mathbb{R}^N \setminus 0 : d\xi \phi = 0\}.
\]

The canonical relation parametrised by $\phi$ is defined as
\[
(2.2) \quad C = \{(y, d_y \phi(x, y, \xi)), (x, -d_x \phi(x, y, \xi)) : (x, y, \xi) \in \Sigma_\phi\},
\]

**Definition 2.5.** Let $X \subset \mathbb{R}^{n_1}$, $Y \subset \mathbb{R}^{n_2}$ be open sets. A Fourier integral operator (FIO) of order $m + N/2 - (n_x + n_y)/4$ is an operator $A : C_0^\infty(X) \rightarrow \mathcal{D}'(Y)$ with Schwartz kernel given by an oscillatory integral of the form
\[
(2.3) \quad K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x, y, \xi)} a(x, y, \xi) d\xi,
\]
where $\phi$ is a clean phase function and $a \in S^m(X \times Y \times \mathbb{R}^N)$ a symbol. The canonical relation of $A$ is the canonical relation of $\phi$ defined in (2.2).

This is a simplified version of the definition of FIO in [2, Section 2.4] or [12, Section 25.2] that is suitable for our purposes since our phase functions are global. For general information about FIOs see [2, 11, 12].

**Definition 2.6.** Let $C \in T^*(Y \times X)$ be the canonical relation associated to the FIO $A : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$. Then we denote $\pi_L$ and $\pi_R$ to be the natural left- and right-projections of $C$, $\pi_L : C \rightarrow T^*Y \setminus 0$ and $\pi_R : C \rightarrow T^*X \setminus 0$.

We have the following result from [12].

**Proposition 2.7.** Let $\dim(X) = \dim(Y)$. Then at any point in $C$:

(i) if one of $\pi_L$ or $\pi_R$ is a local diffeomorphism, then $C$ is a local canonical graph;
(ii) if one of the projections $\pi_R$ or $\pi_L$ is singular at a point in $C$, then so is the other.

The type of the singularity may be different but both projections drop rank on the same set
\[
(2.4) \quad \Sigma = \{(y, \eta; x, \xi) \in C : \det(d\pi_L) = 0\} = \{(y, \eta; x, \xi) \in C : \det(d\pi_R) = 0\}.
\]

If a FIO $F$ satisfies our next definition, then $F^*F$ (or $F^*\phi F$ if $F$ and $F^*$ cannot be composed) is a pseudodifferential operator [4, 17].

**Definition 2.8** (Semi-global Bolker Assumption). Let $F : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$ be a FIO with canonical relation $C$ then $F$ (or $C$) satisfies the semi-global Bolker Assumption if the natural projection $\pi_Y : C \rightarrow T^*(Y)$ is an injective immersion.
3. A TORIC SECTION TRANSFORM

In this section we recall some notation and definitions and introduce a toric section transform which models the intensity of scattered radiation described by the acquisition geometry in figure 1. We later go on to prove our main results which prove the injectivity of the toric section transform on the set of smooth functions compactly supported on a hollow ball and describe microlocally the expected artefacts in a reconstruction from toric section integral data.

For $r > 0$, let $B_r$ be the open disk centered at the origin of radius $r$ and let $B = B_1$ denote the open unit disk. For $X$ an open subset of $\mathbb{R}^n$, let $\mathcal{D}'(X)$ denote the vector space of distributions on $X$, and let $\mathcal{E}'(X)$ denote the vector space of distributions with compact support contained in $X$.

Let us parametrise points on the unit circle, $\theta \in S^1$ as $\theta = \theta(\alpha) = (\cos \alpha, \sin \alpha)$, for $\alpha \in [0, 2\pi]$, and let $\theta_\alpha = \frac{d\theta}{d\alpha}$ be the unit vector $\pi/2$ radians CCW from $\theta$. When the choice of $\alpha$ is understood, then we will write $\theta$ for $\theta(\alpha)$.

Let $(r, \alpha) \in Y := (2, \infty) \times [0, 2\pi)$. To define the toric section, we first define two circular arcs and their centers. For $(r, \alpha) \in Y$ define

$$s = \sqrt{r^2 - 4}, \quad c_1 = c_1(r, \alpha) = \theta(\alpha) + s\theta_\alpha(\alpha), \quad c_2 = c_2(r, \alpha) = \theta(\alpha) - s\theta_\alpha(\alpha)$$

$$C_1 = C_1(r, \alpha) = \{x \in \mathbb{R}^2 : x \cdot \theta_\alpha > 0, \ |x - c_1(r, \alpha)|^2 - r^2 = 0\},$$

$$C_2 = C_2(r, \alpha) = \{y \in \mathbb{R}^2 : y \cdot \theta_\alpha < 0, \ |y - c_2(r, \alpha)|^2 - r^2 = 0\}.$$  

When the choice of $(r, \alpha)$ is understood, we will refer to the arcs as $C_j$ and their centers as $c_j$ for $j \in \{1, 2\}$.

The toric transform integrates functions on $B$ over the toric sections, $C_1(r, \alpha) \cup C_2(r, \alpha)$: let $f \in C_0^\infty(B)$ represent the charge density in the plane. Then, we define the circle transforms

$$\mathcal{T}_1 f(r, \alpha) = \int_{C_1} f ds, \quad \mathcal{T}_2 f(r, \alpha) = \int_{C_2} f ds.$$  

and the toric section transform

$$\mathcal{T} f(r, \alpha) = \int_{C_1 \cup C_2} f ds = \mathcal{T}_1(f)(r, \alpha) + \mathcal{T}_2(f)(r, \alpha)$$  

where $ds$ denotes the arc element on a circle.

**Remark 3.1.** Let $j = 1, 2$. The adjoint, $\mathcal{T}_j^\dagger$, of $\mathcal{T}_j$ is defined by duality on distributions. For $g \in \mathcal{D}(Y)$ and $x \in \mathbb{R}^2 \setminus \mathbf{0}$, $\mathcal{T}^\dagger g(x)$ is a weighted integral of $g$ over all toric sections through $x$. Since there are no toric sections intersecting points outside of $B_3$, we assume $x \in B_3$. We also must exclude $x = \mathbf{0}$ since there are no toric sections through $\mathbf{0}$. Toric sections close to $\mathbf{0}$ have values of $r \approx \infty$.

Furthermore, for fixed $x \in B_3 \setminus B$, the values of $\alpha$ such that $x \in C_j(r, \alpha)$ (for some $r$) is an interval, and the value of $r$ at the endpoints is $r = 2$.

Since the set of toric sections is unbounded, $\mathcal{T}^\dagger$ must be defined on distributions of compact support.

To deal with all of these inconveniences, we define a modified adjoint. Let $\varphi : (2, \infty) \to \mathbb{R}$ be smooth and with compact support in $(2, M)$ for some $M > 2$. One can also assume $0 \leq \varphi \leq 1$ and $\varphi = 1$ on most of $(2, M)$. We define the cutoff-adjoint $\mathcal{T}_j^* : \mathcal{D}'(Y) \to \mathcal{D}'(B_3)$. For $g \in \mathcal{D}'(Y)$,

$$\mathcal{T}_j^* g = \mathcal{T}_j^\dagger(\varphi g), \quad \mathcal{T}^* = \mathcal{T}_1^* + \mathcal{T}_2^*.$$  

Let $\rho_{\text{min}} = M - \sqrt{M^2 - 3}$. Then, $T^*g(x) = 0$ for $x \in B_{\rho_{\text{min}}} \setminus 0$. This is true because $\rho_{\text{min}}$ is the closest distance of the arcs $C_1(r, \alpha)$ and $C_2(r, \alpha)$ get to the origin for all $(r, \alpha) \in (2, M) \times [0, 2\pi]$. Therefore, we define $T^*g(0) = 0$ and $T^*g$ is smooth near 0. This also means for $f \in \mathcal{E}'(B)$ that $T^*f(x) = 0$ if $x \in B_{\rho_{\text{min}}}$.

3.1. Microlocal Properties of $T_1$ and $T$. Since we do not consider the points of intersection of the arcs $C_1$ and $C_2$ (since $B$ is open), we can consider the microlocal properties of the circle transforms $T_1$ and $T_2$ separately. Let us consider $T_2$. Let $Y = [0, 2\pi] \times (2, \infty)$. When considering functions and distributions on $Y$, we use the standard identification of $[0, 2\pi]$ with the unit circle $S^1$, $\alpha \mapsto \theta(\alpha) = (\cos(\alpha), \sin(\alpha))$.

We first show $T_1$ and $T_2$ are FIO and then prove that they satisfy the Bolker Assumption.

**Proposition 3.2.** $T_1$ and $T_2$ are FIO and their canonical relations are

$$
C_1 = \left\{ (r, \alpha, -2\sigma x \cdot (\theta_\alpha - s\theta), -\frac{2\sigma r}{s}(x \cdot \theta_\alpha); x, -2\sigma(x - c_1(r, \alpha))) : (r, \alpha) \in Y, \sigma \in \mathbb{R} \setminus 0, x \in C_1(r, \alpha) \cap B \right\}.
$$

(3.4)

$$
C_2 = \left\{ (\alpha, r, -2\sigma y \cdot (\theta_\alpha + s\theta), \frac{2\sigma r}{s}(y \cdot \theta_\alpha); y, -2\sigma(y - c_2(r, \alpha))) : (r, \alpha) \in Y, \sigma \in \mathbb{R} \setminus \{0\}, y \in C_2(r, \alpha) \cap B \right\}.
$$

For $j = 1, 2$, we let $\tilde{C}_j$ be defined as $C_j$ except that $x$ or $y$ is not restricted to be in $B$ and we let $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$.

**Proof.** We briefly explain why $T_2$ is a FIO and we calculate its canonical relation. Let $Z = \{(r, \alpha, y) \in Y \times B : |y - c_2(r, \alpha)|^2 - r^2 = 0\}$. From calculations in [4, 17] the Schwartz kernel of $T_2$ is integration over $Z$ and so the this Schwartz kernel is a Fourier integral distribution with phase function $\phi_2(y, r, \alpha, \sigma) = \sigma(|y - c_2(r, \alpha)|^2 - r^2)$. This is true because, for functions supported in $B$, $T_2$ can be viewed as integrating on the full circle defined by $|y - c_2(r, \alpha)|^2 - r^2 = 0$.

Using Definition 2.4 one sees that the canonical relation of $T_2$ is given the expression in (3.4). One can easily check that the projections $\pi_L(C_2)$ and $\pi_R(C_2)$ do not map to the zero section so $T_2: \mathcal{E}'(B) \rightarrow \mathcal{D}'(Y)$ [9]. The proof for $T_1$ is similar but its phase function is $\phi_1(x, r, \alpha, \sigma) = \sigma(|x - c_1(r, \alpha)|^2 - r^2)$.

We next have the theorem:

**Theorem 3.3.** For $j = 1, 2$, the left projection $\pi_L : \tilde{C}_j \rightarrow T^*(Y)$ is an injective immersion. Therefore, $\pi_L : C_j \rightarrow T^*(Y)$ is an injective immersion and so $T_j$ satisfies the semi-global Bolker Assumption (Definition 2.8).

**Proof.** We will prove this theorem for $T_2$ and the proof for $T_1$ is completely analogous. We first show that $\pi_L$ is an immersion.

As noted above, if $\alpha$ is known, then we let $\theta = \theta(\alpha)$ and $\theta_\alpha = (-\sin \alpha, \cos \alpha)$. For bookkeeping reasons, if $\beta \in [0, 2\pi]$, the vector in $S^1$ corresponding to $\beta$ will be denoted $\psi = (\cos \beta, \sin \beta)$ and we let $\psi_\beta = (-\sin \beta, \cos \beta)$ be the unit vector $\pi/2$ radians CCW from $\psi$. This allows us to parametrise points on $C_2(r, \alpha)$ by

$$
y = y(r, \alpha, \beta) = c_2 + r\psi = c_2(r, \alpha) + r(\cos \beta, \sin \beta),
$$

for $\beta$ in an open interval of $[0, 2\pi]$. Then,

$$
(r, \alpha, \beta, \sigma) \mapsto \lambda_2(r, \alpha, \beta, \sigma) := (r, \alpha, \sigma \delta^x_\alpha \phi_2, \sigma \delta^x_\phi_2; y(r, \alpha, \beta), -\sigma \delta^y_\phi_2) \in C_2
$$

(3.6)
gives coordinates on the canonical relation \( C_2 \). Using these coordinates and after simplification, the map \( \pi_L \) is given by
\[
(3.7) \quad \pi_L(\lambda(r, \alpha, \beta, \sigma)) = \left( r, \alpha, -2\sigma r \psi \cdot (\theta_\alpha + s \theta), \frac{2\sigma r}{s} (-s + r \psi \cdot \theta_\alpha) \right)
\]
and
\[
(3.8) \quad D\pi_L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\sigma_1 & \sigma_2 & \frac{2\sigma r}{s} (\psi \cdot \theta_\alpha) & \frac{2\sigma r}{s} (-\frac{s}{r} + \psi \cdot \theta_\alpha) \\
\sigma_4 & \sigma_2 & \frac{2\sigma r}{s} (\psi \cdot \theta_\alpha) & \frac{2\sigma r}{s} (-\frac{s}{r} + \psi \cdot \theta_\alpha)
\end{pmatrix}.
\]

It follows that
\[
(3.9) \quad \det D\pi_L = -\frac{4r^3\sigma}{s} \det \left( \begin{array}{cc}
\psi_\beta \cdot (\theta_\alpha + s \theta) & \psi \cdot (\theta_\alpha + s \theta) \\
\psi_\beta \cdot \theta_\alpha & -\frac{s}{r} + \psi \cdot \theta_\alpha
\end{array} \right)
\]
\[
= -\frac{4r^3\sigma}{s} \left( -\frac{s}{r} \psi_\beta \cdot (\theta_\alpha + s \theta) + (\psi \cdot \theta_\alpha)(\psi_\beta \cdot \theta_\alpha + s \psi_\beta \cdot \theta) - (\psi_\beta \cdot \theta_\alpha)(\psi \cdot \theta_\alpha + s \psi \cdot \theta) \right)
\]
\[
= -4r^3\sigma \left( \frac{1}{r} \psi_\beta \cdot (\theta_\alpha + s \theta) + (\psi \cdot \theta_\alpha)(\psi_\beta \cdot \theta) - (\psi_\beta \cdot \theta_\alpha)(\psi \cdot \theta) \right)
\]
\[
= 4r^3\sigma \left( \frac{1}{r} (\psi \cdot c_2) + ((\psi \cdot \theta_\alpha)^2 + (\psi \cdot \theta)^2) \right)
\]
\[
= 4r^3\sigma \left( \frac{1}{r} (\psi \cdot c_2) + 1 \right),
\]
where to go from step 3 to 4 above we have used the identities \( \psi_\beta \cdot \theta_\alpha = \psi \cdot \theta \) and \( \psi_\beta \cdot \theta = -\psi \cdot \theta_\alpha \). Let us assume \( \det D\pi_L = 0 \). Then \( \psi \cdot c_2 = -r \). But \( |\psi \cdot c_2| \leq |c_2| = \sqrt{r^2 - 3} < r \) and we have a contradiction. Note that this contradiction holds for all \( y \in C_2(r, \alpha) \), not just those in \( B \). Therefore, the map \( \pi_L : \tilde{C}_2 \to T^*(Y) \) is an immersion.

We next show the injectivity of the left projection \( \pi_L \) through an analysis of the canonical relations of \( \mathcal{T}_2 \). Let \( (r, \alpha, \eta, y_1, \xi) \) and \( y_1, y_2 \) be two points in \( C_2(\alpha) \) and \( \xi \) in \( \mathbb{R}^2 \setminus 0 \) such that \( (r, \alpha, \eta, y_1, \xi) \) and \( (r, \alpha, \eta, y_2, \xi) \) are both in \( C_2 \). We show \( (y_1, \xi) = (y_2, \xi) \). By equating the terms for \( \eta \) in the expression for \( C_2 \), (3.4), one sees, for some \( \sigma_1 \) and \( \sigma_2 \), that
\[
(3.10) \quad \eta = \left( \frac{-2\sigma_1 y_1 \cdot (\theta_\alpha + s \theta)}{2\sigma_1 r}, \frac{2\sigma_2 y_2 \cdot (\theta_\alpha + s \theta)}{2\sigma_2 r} \right) = \left( \frac{-2\sigma_2 y_2 \cdot (\theta_\alpha + s \theta)}{2\sigma_2 r}, \frac{2\sigma_1 y_1 \cdot (\theta_\alpha + s \theta)}{2\sigma_1 r} \right)
\]
where \( s = \sqrt{r^2 - 4} \). Since \( y_j \cdot \theta_\alpha < 0 \), the bottom equation in (3.10) shows that \( \nu = \sigma_1/\sigma_2 > 0 \). In addition,
\[
(3.11) \quad \frac{2\sigma_1 r}{s} (y_1 \cdot \theta_\alpha) = \frac{2\sigma_2 r}{s} (y_2 \cdot \theta_\alpha) \implies (\sigma_1 y_1 - \sigma_2 y_2) \cdot \theta_\alpha = 0
\]
and
\[
(3.12) \quad -2\sigma_1 y_1 \cdot (\theta_\alpha + s \theta) = -2\sigma_2 y_2 \cdot (\theta_\alpha + s \theta) \implies (\sigma_1 y_1 - \sigma_2 y_2) \cdot \theta = 0.
\]
Hence \( \sigma_1 y_1 - \sigma_2 y_2 = 0 \) or \( y_2 = \nu y_1 \) where \( \nu = \frac{\sigma_1}{\sigma_2} > 0 \). Given that any ray through origin intersects the curve \( C_2 \) at most once and \( y_1, y_2 \in C_2 \), it follows that \( \sigma_1 = \sigma_2 \) and \( y_1 = y_2 \). This finishes the proof for \( \mathcal{T}_2 \). Note that this proof is valid for any \( y_1 \) and \( y_2 \) in \( C_2 \), not just for those in \( B \). In other words, \( \pi_L : \tilde{C}_2 \to T^*(Y) \) is also injective, so \( \pi_L : \tilde{C}_2 \to T^*(Y) \) is an injective immersion.
As already noted, the proof for $T_i$ is similar, and it uses the following coordinate maps
\begin{align}
(x, y) &= x(r, \alpha, \beta) = c_1 + r \psi = c_1(r, \alpha) + r(\cos \beta, \sin \beta), \quad \beta \in [0, 2\pi], \\
r, \alpha, \beta, \sigma, \lambda, \pi, \eta &= \lambda_1(r, \alpha, \beta, \sigma) := (r, \alpha, \sigma d_{\alpha} \phi_1, \sigma d_{\alpha} \phi_1; x(r, \alpha, \beta), -\sigma d_{\alpha} \phi_1) \in C_1,
\end{align}
however in this case, $\beta$ is in an open interval $[\pi, \pi]$.
\hfill $\square$

Let $C = C_1 \cup C_2$. Because $C_1 \cap C_2 = \emptyset$, $C$ is an embedded Lagrangian manifold and since $T = T_1 \cup T_2$, $T$ is a FIO with canonical relation $C$. We now have our main theorem which shows that the canonical relation $C$ is $2\to1$ in a specific sense. We give explicit expressions for the expected artefacts in a reconstruction using $T^*T$ that are caused by this $2\to1$ map.

**Theorem 3.4.** The projection $\pi_L : C \to T^*(Y)$ is two-to-one in the following sense. Let $\lambda = (r, \alpha, \eta) \in \pi_L(C)$. Then, there is at least one point $(w, \xi) \in B \times (\mathbb{R}^2 \setminus 0)$ such that $\pi_L(\lambda, (w, \xi))$. Necessarily, $w$ is either in $C_1(r, \alpha)$ or in $C_2(r, \alpha)$. Assume $w \in C_1$. Then, there is a $y \in C_2$ and $\xi \in \mathbb{R}^2 \setminus 0$ such that $\lambda = \pi_L(\lambda, (y, \xi))$. The point $y$ is given by (3.17). If $w \in C_2$, then its corresponding point in $C_1$ is given by (3.18).

Let $T^*$ be the modified dual operator in Remark 3.1. The canonical relation of $T^*T$ is of the form $\Delta \cup \Lambda_1 \cup \Lambda_2$, where $\Delta$ is the diagonal in $T^*X \times T^*X$ and $\Lambda_1 = \widetilde{C}_1 \circ C_2$ and $\Lambda_2 = \widetilde{C}_2 \circ C_1$ are associated to reconstruction artefacts.

Let $f$ be a distribution supported in $B$. If $(w, \xi) \in WF(f)$ and $\xi \cdot w \neq 0$, then two artifacts are generated in $T^*T f$ associated with $(w, \xi)$. The base point of the one generated by $\Lambda_1$ is given by (3.24) where $r$ is defined by (3.20) and $\alpha$ is solved from (3.23) and the base point of the artifact caused by $\Lambda_2$ is given by (3.22) where $r$ is defined by (3.20) and $\alpha$ is given by solving (3.21).

**Proof.** Let $(r, \alpha, \eta) \in \pi_L(C)$, then there is an $(w, \xi) \in B \times (\mathbb{R}^2 \setminus 0)$ such that $(r, \alpha, \eta, w, \xi) \in \mathcal{C}$. Either $w \in C_1(r, \alpha)$ or $w \in C_2(r, \alpha)$, and this is determined by $(r, \alpha)$. At the end of this part of the proof, we will outline what to do if $w \in C_2(r, \alpha)$.

We assume $w \in C_1(r, \alpha)$ and, for this part of the proof—in which $w \in C_1$—we let $x = w$. Assume there is another point in $\mathcal{C}$ that maps to $(r, \alpha, \eta)$ under $\pi_L$. That point must be on $\widetilde{C}_2$ and it must be unique since $\pi_L : \widetilde{C}_j \to T^*(Y)$ is injective for $j = 1, 2$ by Theorem 3.3. Let $(y, \xi)$ be chosen so $y \in C_2(r, \alpha)$ and $(r, \alpha, \eta; y, \xi)$ is the preimage in $\widetilde{C}_2$ of $(r, \alpha, \eta)$. Comparing the $\eta$ term of the expressions (3.4) for $C_1$ and $C_2$, we see there are numbers $\sigma_1$ and $\sigma_2$ such that
\begin{equation}
\eta = \left( \begin{array}{c}
-2\sigma_1 x \cdot (\theta_\alpha - s\theta) \\
-2\sigma_2 y \cdot (\theta_\alpha + s\theta)
\end{array} \right) = \left( \begin{array}{c}
2s x \cdot \theta_\alpha \\
2s y \cdot \theta_\alpha
\end{array} \right),
\end{equation}
This implies that $\sigma_1 (x \cdot \theta_\alpha) = -\sigma_2 (y \cdot \theta_\alpha)$. Since $x \cdot \theta_\alpha$ and $y \cdot \theta_\alpha$ have opposite signs, $\sigma_1$ and $\sigma_2$ have the same sign. Let $\nu = \sigma_1 / \sigma_2$, then $\nu > 0$ and if we solve (3.15) for $y$, we see
\begin{equation}
y = \nu \left( (-x \cdot \theta_\alpha) \theta_\alpha + \left( \frac{2}{s} x \cdot \theta_\alpha - x \cdot \theta \right) \theta \right) \quad \text{for some } \nu > 0.
\end{equation}

Equivalently we can write the above as
\begin{equation}
y = \nu[\theta_\alpha, \theta] \left[ \frac{2}{s} \theta_\alpha \cdot \theta \right] x, \quad (C_1 \to C_2).
\end{equation}

Given $r, \alpha$, and $x$, this equation describes the point $y$ that is the base point of the preimage in $\widetilde{C}_2$ of $(r, \alpha, \eta)$.

Equation (3.17) for arbitrary $\nu > 0$ describes a ray starting at $0$. Because the circle containing $C_2(r, \alpha)$ encloses $0$, this ray intersects the circle at a unique point. Since any
point \( y' \) on this ray satisfies \( y' \cdot \theta_\alpha < 0 \), the unique point on the circle is on \( C_2(r, \alpha) \). If \( w = x \in C_1 \), then this proves that \( \pi_L \) is two-to-one as described in the theorem.

To prove the statement about \( \pi_L \) being two-to-one if the point \( w \) at the start of the proof is in \( C_2(r, \alpha) \) then one goes through the same proof but solves for \( x \) in terms of \( y \) and replace \( y \) by \( w \) in (3.15) to get

\[
(3.18) \quad x = \frac{1}{\nu} \left[ \theta_\alpha, \theta \right] \left[ \begin{array}{c} -\theta T \alpha \\ -\frac{2}{s} \theta T \alpha \end{array} \right] w, \quad (C_2 \to C_1).
\]

Given \( r, \alpha \) and \( w \), this equation describes the point \( x \) that is the base point of the preimage in \( \tilde{C}_1 \) of \( (r, \alpha, \eta) \).

To describe explicitly the artefacts which occur due to an application of the normal operator \( T^*T \), let us consider the canonical relation \( \tilde{C} \circ \tilde{C} \). We have the expansion

\[
(3.19) \quad \tilde{C} \circ C = (\tilde{C}_1 \cup C_2) \circ (\tilde{C}_1 \cup C_2)
\]

\[
= (\tilde{C}_1 \cup C_1) \cup (\tilde{C}_2 \cup C_2) \cup (\tilde{C}_1 \cup C_2) \cup (\tilde{C}_2 \cup C_1)
\]

\[
\subset \Delta \cup \Lambda_1 \cup \Lambda_2,
\]

where \( \Lambda_1 = \tilde{C}_1 \cup C_2 \) and \( \Lambda_2 = \tilde{C}_2 \cup C_1 \). Note that \( \tilde{C}_j \circ C_j \subset \Delta \) for \( j = 1, 2 \) because \( \tilde{C}_j \) satisfies the Bolker Assumption.

Let \( (w, \xi) \in T^*(B) \) be such that \( w \cdot \xi \neq 0 \) and let \( \xi' = \xi/|\xi| \). We now calculate the \( (r, \theta) \) for which the circle \( C_1 \) intersects \( w \) normal to \( \xi \), explicitly in terms of \( (w, \xi) \). For \( w \in C_1 \) we know \( c_1 = w - r\xi' \). Therefore

\[
|c_1|^2 = r^2 - 3 = |w - r\xi'|^2 = |w|^2 - 2r \cdot \xi' + r^2
\]

and it follows that

\[
(3.20) \quad r = \frac{|w|^2 + 3}{2(w \cdot \xi')},
\]

Also, to get \( (r, \theta) \) explicitly in terms of \( (w, \xi') \),

\[
(3.21) \quad \left( \begin{array}{cc} 1 & s \\ -s & 1 \end{array} \right) \theta = w - r\xi'.
\]

To check that \( \theta \) is a unit vector, note that

\[
|\theta| = \frac{1}{1 + s^2} \left| \left( \begin{array}{cc} 1 & -s \\ s & 1 \end{array} \right) (x - r\xi') \right| = \frac{|w - r\xi'| \sqrt{1 + s^2}}{1 + s^2} = 1,
\]

as \( |w - r\xi'| = |c_1| = \sqrt{1 + s^2} \). Once \( (r, \theta) \) are known, the artefact \( y \) induced by \( \Lambda_2 \) is given by equation (3.17)

\[
(3.22) \quad y = \nu[\theta_\alpha, \theta] \left[ \begin{array}{c} -\theta T \alpha \\ \frac{2}{s} \theta T \alpha \end{array} \right] w
\]

where \( \nu > 0 \) is such that \( y \in C_2 \). This point \( y \) is the base point of the artefact corresponding to \( (w, \xi) \) that is added by \( \Lambda_1 \).

Similarly we can express the \( (r, \theta) \) for which the circle \( C_2 \) intersects \( w \) normal to \( \xi \), explicitly in terms of \( (w, \xi) \). When \( w \in C_2 \), we know \( c_2 = w - r\xi' \). Hence the calculation for \( r \) is the same as (3.20) and

\[
(3.23) \quad \left( \begin{array}{cc} 1 & -s \\ s & 1 \end{array} \right) \theta = w - r\xi',
\]
and hence the artefact $x$ induced by $\Lambda_1$ is given by (3.18)

$$
(3.24) \quad x = \frac{1}{\nu} \big[ \theta_\alpha, \theta \big] \left[ -\theta_T^{\alpha} \right] w,
$$

where $\nu$ is chosen so $x \in C_1$. Then, $x$ is the base point of the artefact in $\Lambda_1$ caused by $(w, \xi)$.

\[ \square \]

### 3.2. Injectivity

Here we prove the injectivity of the toric section transform $T$ on the set of smooth functions compactly supported on a hollow ball. For $0 \leq \epsilon_1 < \epsilon_2$ we let $B_{\epsilon_1, \epsilon_2} = \{ x \in \mathbb{R}^2 : \epsilon_1 < |x| < \epsilon_2 \}$ denote the open annulus in $\mathbb{R}^2$ with inner radius $\epsilon_1$ and outer radius $\epsilon_2$.

In this section we write points in $\mathbb{R}^2$ in polar coordinates $(\rho, \alpha) \mapsto \rho \theta(\alpha) = \rho(\cos(\alpha), \sin(\alpha))$. For an integrable function $F(\rho, \alpha)$ and $l \in \mathbb{Z}$, we define the $l^{th}$ polar Fourier coefficient of $f$ to be

$$
(3.25) \quad F_l(\rho) = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} F(\rho, \alpha) e^{-il\alpha} \, d\alpha.
$$

Let $t = \sqrt{t^2 - 3}$ and let $\alpha(t) = \cos^{-1} \frac{1}{t}$. Then we can parametrise the set of points on the toric section in polar coordinates

$$
(3.26) \quad \rho = \sqrt{t^2 \cos^2 \varphi + 3} - t \cos \varphi, \quad -\alpha(t) \leq \varphi \leq \alpha(t), \quad t \geq 1
$$

and it follows that

$$
(3.27) \quad T f(t, \alpha) = \int_{-\alpha(t)}^{\alpha(t)} \sqrt{\rho^2 + \left( \frac{\partial \rho}{\partial \varphi} \right)^2} \left( F(\rho, \alpha + \alpha(t) + \varphi) + F(\rho, \alpha - \alpha(t) + \varphi) \right) |_{\rho=\sqrt{t^2 \cos^2 \varphi + 3} - t \cos \varphi} \, d\varphi,
$$

where $F(\rho, \alpha) = f(\rho \theta(\alpha))$ is the polar form of $f$. We now have our second main theorem which follows similar ideas to that of Cormack [1] to prove the injectivity of $T$ for $f \in C^\infty_0(B_{\epsilon,1})$, where $0 < \epsilon < 1$.

**Theorem 3.5.** The toric section transform $T : C^\infty_0(B_{\epsilon,1}) \to C^\infty(Y)$, where $0 < \epsilon < 1$ and $Y = (2, \infty) \times [0, 2\pi]$, is injective.

**Proof.** After exploiting the rotational invariance of the transform (3.26) we have

$$
(3.28) \quad T f_l(t) = T_{|l|} \left( \frac{1}{t} \right) \int_{-\alpha(t)}^{\alpha(t)} \sqrt{\rho^2 + \left( \frac{\partial \rho}{\partial \varphi} \right)^2} F_l(\rho) e^{-il\varphi} |_{\rho=\sqrt{t^2 \cos^2 \varphi + 3} - t \cos \varphi} \, d\varphi,
$$

where

$$
(3.29) \quad T f_l(t) = \frac{1}{2\pi} \int_0^{2\pi} T f(t, \alpha) e^{-il\alpha} \, d\alpha,
$$

and $T_{|l|}$ is Chebyshev polynomial of the first kind of order $|l|$.

The arc length measure on the circle is

$$
(3.30) \quad ds = \sqrt{\rho^2 + \left( \frac{\partial \rho}{\partial \varphi} \right)^2} \, d\varphi = \rho \sqrt{t^2 + 3 \cos^2 \varphi + 3} \, d\varphi = \rho \left( 1 - \frac{t \cos \varphi}{\sqrt{t^2 \cos^2 \varphi + 3}} \right) \, d\varphi,
$$

where

$$
(3.31) \quad \phi \mapsto \psi = (2, \infty) \times [0, 2\pi],
$$

and

$$
(3.32) \quad T f_l(t) = \frac{1}{2\pi} \int_0^{2\pi} T f(t, \alpha) e^{-il\alpha} \, d\alpha.
$$
and using the symmetry of equation (3.27) in \( \phi \) about \( \phi = 0 \) we have

\[
\frac{Tf_i(t)}{2r} = T[|l| \left( \frac{1}{t} \right) \int_0^{\alpha(t)} \left( 1 - \frac{t \cos \phi}{\sqrt{t^2 \cos^2 \phi + 3}} \right) F_i(\rho) \cos(l\phi) \mid_{\rho=\sqrt{\rho^2 \cos^2 \phi + 3 - t \cos \phi}} \ d\rho
\]

\[
= T[|l| \left( \frac{1}{t} \right) \int_0^{\alpha(t)} \tilde{F}_i(t \cos \phi) \cos(l\phi) d\phi,
\]

where \( \tilde{f} \in C_0^\infty(B_1, \mathbb{R}^2) \) is defined as

\[
\tilde{f}(x) = \left( 1 - \frac{|x|}{\sqrt{|x|^2 + 3}} \right) f \left( (\sqrt{|x|^2 + 3} - |x|), \frac{x}{|x|} \right)
\]

and \( \tilde{F}(\rho, \alpha) = \tilde{f}(\rho \theta(\alpha)) \) is the polar form of \( \tilde{f} \).

After making the substitution \( \rho = s \cos \phi \) we have

\[
\frac{Tf_i(t)}{2r} = T[|l| \left( \frac{1}{t} \right) \int_1^t \tilde{F}_i(\rho)T[|l| \left( \frac{\rho}{t} \right) \frac{1}{\sqrt{t^2 - \rho^2}} \ d\rho.
\]

The Chebyshev polynomials \( T[|l| \) have \( l \) real roots, say \( x_1, \ldots, x_{|l|}, \) which satisfy the inequality \(-1 < x_1 < x_2, \ldots, < x_{|l|} < 1\). Let \( 1 < t < \frac{1}{x_{|l|}} \). Since \( \tilde{F}_i \in C_0^\infty([1, \frac{3-x^2}{2t}]) \), the function \( g_l \) defined by

\[
g_l(t) = \int_1^t \tilde{F}_i(\rho)T[|l| \left( \frac{\rho}{t} \right) \frac{1}{\sqrt{t^2 - \rho^2}} \ d\rho
\]

is smooth on \([1, \frac{3-x^2}{2t}]\).

Let \( Tf_l = 0 \). Then \( g_l = 0 \) almost everywhere (it is zero except for on the finite set \( \bigcup_{i=1}^{|l|} \{1/x_i\} \)). Hence, as \( g_l \) is continuous, \( g_l = 0 \). So we have

\[
\int_1^t \tilde{F}_i(\rho)T[|l| \left( \frac{\rho}{t} \right) \frac{1}{\sqrt{t^2 - \rho^2}} \ d\rho = 0
\]

for all \( 1 < t < \infty \). Now to prove injectivity we follow a similar argument to that of Cormack in [1, page 2724].

Let \( 1 < t < \frac{1}{x_i} \). Then \( x_i < 1/t < \rho/t < 1 \). So \( T[|l|/(\rho/t) / \sqrt{t^2 - \rho^2} \geq 0 \) on the bounds of integration and the above integrand (3.34) is positive apart from \( \tilde{F}_i \). Hence, as \( \tilde{F}_i \) is continuous, this implies \( \tilde{F}_i = 0 \) for \( 1 < \rho < \frac{1}{x_i} \). Now we need only consider

\[
\int_{\frac{1}{x_i}}^t \tilde{F}_i(\rho)T[|l| \left( \frac{\rho}{t} \right) \frac{1}{\sqrt{t^2 - \rho^2}} \ d\rho = 0.
\]

Let \( \frac{1}{x_i} < t < \frac{1}{x_i} \). Then \( x_i < 1/(x_i t) < \rho/t < 1 \). Hence \( T[|l|/(\rho/t) \geq 0 \) and it follows from the continuity of \( \tilde{F}_i \) that \( \tilde{F}_i = 0 \) for \( 1 < \rho < \frac{1}{x_i} \). Letting \( m \to \infty \), \( 1/x_i^m \to \infty \) as \( x_i < 1 \). Hence the only solution to equation (3.34) is \( \tilde{F}_i = 0 \) for \( 1 < \rho < \infty \). From the definition of \( \tilde{F}_i \) (given by equation (3.31)), it follows that \( F_l = 0 \) for \( 0 < \rho < 1 \).

So far we have shown that the problem of reconstructing a density \( f \in C_0^\infty(B_{x,1}) \) from \( Tf \) is uniquely solvable, and provided explicit expressions for the expected artefacts in the reconstruction. We next go on to demonstrate our theory through a discrete simulation.
4. RECONSTRUCTION ALGORITHM AND RESULTS

Here we present reconstruction algorithms for the reconstruction of two dimensional densities from toric section integral data and demonstrate the artefacts described by the theory in section 3.1.

We take a discrete (algebraic) approach to reconstruction. That is we discretize the operator $\mathcal{T}$ on a pixel grid (see figure 9) and solve the set of linear equations

$$\|Ax - b\|^2 + \lambda^2 \mathcal{G}(x),$$

where $A$ is the discrete form of $\mathcal{T}$ (each row of $A$ is the vectorized form of a binary image as shown in figure 9) and $\mathcal{G}(x)$ is a regularisation penalty (e.g. $\mathcal{G}(x) = \|x\|^2_2$ (Tikhonov) or $\mathcal{G}(x) = \sum_i |x_i - x_{i-1}|$ (TV)), with regularisation parameter $\lambda$. Here $x$ represents the vectorized form of the density image (which is to be reconstructed) and $b$ (our data) represents the Compton scattered intensity.

To simulate noisy data we take a vectorized density image $x$ (such as those presented in figure 4) we add a Gaussian random noise

$$b = Ax + \epsilon \times g \frac{\|Ax\|_2}{\sqrt{n}},$$

where $g$ is a pseudo random vector of samples drawn from a standard normal distribution and $n$ is the number of entries in $b$. Here $\epsilon$ denotes the noise level in the sense that $\frac{\|b - Ax\|_2}{\|Ax\|_2} \approx \epsilon$ for $n$ large enough. Throughout the simulations presented here we simulate toric section integral data for rotation angles $\alpha \in \{\frac{j\pi}{180} : 1 \leq j \leq 360\}$ and for circle radii $r \in \{\frac{j^2 + 200^2}{2j} : 1 \leq j \leq 199\}$, where the pixel grid size is 200–200. So $n = 360 \times 199 = 71640$ and $A$ has $200^2$ columns.

![Predicted and observed artefacts from reconstructing a delta function far from the origin by backprojection.](image)

**Figure 2.** Predicted and observed artefacts from reconstructing a delta function far from the origin by backprojection.

To simulate the artefacts implied by the theory presented in section 3.1, let us consider the reconstruction of a delta function by (unfiltered) backprojection. That is by an application of the normal operator $\mathcal{T}^* \mathcal{T} \delta$, where $\delta$ has its support in the unit ball. To describe the artefacts implied by $\Lambda_1 = \tilde{\mathcal{C}}_2 \circ \tilde{\mathcal{C}}_1$ and $\Lambda_2 = \tilde{\mathcal{C}}_1 \circ \tilde{\mathcal{C}}_2$ when $f = \delta$ (so here $f$ is
non zero only at a single point and its wavefront set lies in all directions), let us consider a point \( x = |x|(-1, 0) \) on the \( x \) axis. Then equation (3.24) becomes

\[
y = \left( 1 + \frac{2}{s} \sin \alpha \cos \alpha, \frac{2}{s} \sin^2 \alpha \right)^T
\]

up to scaling. Similarly for \( y = |y|(-1, 0) \) equation (3.22) becomes

\[
x = \left( 1 - \frac{2}{s} \sin \alpha \cos \alpha, -\frac{2}{s} \sin^2 \alpha \right)^T,
\]

again up to scaling. Let us define \( \psi_1 : [-\pi, 0] \rightarrow \text{sg}(\mathbb{R}^2) \) and \( \psi_2 : [0, \pi] \rightarrow \text{sg}(\mathbb{R}^2) \) as

\[
\psi_1(\alpha) = \left\{ \nu \left( 1 + \frac{2}{s} \sin \alpha \cos \alpha, \frac{2}{s} \sin^2 \alpha \right) : \nu \in \mathbb{R} \right\} \cap C_2 \cap \{x \cdot \theta_\alpha > 0\}
\]

and

\[
\psi_2(\alpha) = \left\{ \nu \left( 1 - \frac{2}{s} \sin \alpha \cos \alpha, -\frac{2}{s} \sin^2 \alpha \right) : \nu \in \mathbb{R} \right\} \cap C_1 \cap \{x \cdot \theta_\alpha < 0\}.
\]

where \( \text{sg}(\mathbb{R}^2) \) denotes the set of singleton subsets of \( \mathbb{R}^2 \). Also

\[
s = \left| \frac{3 - |x|^2 + 2(x \cdot \theta)}{2(x \cdot \theta)} \right|,
\]

to get \( s \) in terms of \( x \) and a rotation \( \alpha \). Then \( \psi_1([-\pi, 0]) \) and \( \psi_2([0, \pi]) \) are the set of artefacts in the plane associated to \( \Lambda_1 \) and \( \Lambda_2 \) respectively. Note that we need only consider the domain \([-\pi, 0] \) for \( \psi_1 \) as the circle \( C_1 \) does not intersect \( x = |x|(-1, 0) \) for any \( \alpha \in (0, \pi) \), and conversely for \( \psi_2 \). It is clear that \( \psi_1([-\pi, 0]) = P \psi_2([0, \pi]) \), where \( P \) denotes a reflection in the line \( \{tx : t \in \mathbb{R}\} \) (or the \( x \) axis in this case). Hence the artefacts associated to \( \Lambda_1 \) are those associated to \( \Lambda_2 \) but reflected in the line \( \{tx : t \in \mathbb{R}\} \), for a given \( x \in \mathbb{R}^2 \), when \( f \) has singularities at \( x \) in all directions \( \xi \). We can use equations (4.5) and (4.6) to draw curves in the plane where we expect there to be image artefacts. To simulate \( \delta \) discretely we assign a value of 1 to four neighbouring pixels in the unit cube (discretized as a 200–200 grid) and set all other pixel values to zero. Let our discrete delta function be denoted by \( x_\delta \). Then we approximate \( T^*T \delta \approx A^TAx_\delta \). See figures 2 and 3.

**Figure 3.** Predicted and observed artefacts from reconstructing a delta function closer to the origin by backprojection.
where we have shown side by side comparisons of the artefacts predicted by equations (4.5) and (4.6) and the artefacts observed in a reconstruction by backprojection. See also figures 10 and 11 for more simulated artefact curves. The observed artefacts are as predicted by the theory and the images in the left and right hand sides of each figure superimpose exactly. We notice a cardioid curve artefact in the reconstruction which becomes a full cardioid when the delta function lies approximately on the unit circle.

To test our reconstruction techniques, we consider the test phantoms displayed in figure 4, one simple and one complex. The simple phantom consists of a disc with value 2 and a square with value 1. The complex phantom consists of simulated objects of varying density, shape and size with overlapping ellipsoids, and is commonly used to test reconstruction techniques in tomography [5]. See figures 5, 6, 12, 13 for reconstructions of the two test phantoms using the Landweber method and a Conjugate Gradient Least Squares (CGLS) iterative solver [5] with Tikhonov regularisation (varying the regularisation parameter \( \lambda \) manually). In the absence of noise \( (\epsilon = 0) \) there are significant artefacts in the reconstruction using a Landweber approach. CGLS performs well however on both test phantoms. In the presence of added noise \( \epsilon = 0.01 \) and 5% \( (\epsilon = 0.05) \) there are severe artefacts in the reconstruction using a CGLS with Tikhonov approach (see figures 5 and 6), particularly with a higher noise level of 5%. To combat the image artefacts we found that the use of TV regulariser with a heuristic iterative approach (as described in [3]) was effective. Here we apply non–negativity constraints to the optimizer (as we know a–priori that a density is non–negative) and choose the regularisation parameter \( \lambda \) manually. See figures 7 and 8. For a noise level of 1% the artefacts are almost completely removed from the reconstructions (for both the simple and complex phantom) and the image quality is high overall. For a higher noise level of 5% we see a significant reduction in the artefacts and the reconstruction is satisfactory in both cases with a low level of distortion in the image (although there is a higher distortion in the complex phantom reconstruction).

For the application considered in this paper, namely threat detection in airport baggage screening, the removal of image artefacts and an accurate quantitative density estimation are crucial to maintain a satisfactory false positive rate. We will now further compare our results using CGLS with Tikhonov and the iterative solver of [3], in terms of the false positive rate we can expect using both methods. Looking at the reconstructions using

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phantoms.png}
\caption{Simple (left) and complex (right) phantoms.}
\end{figure}
Figure 5. Simple phantom reconstruction using CGLS and Tikhonov as a regulariser, with noise levels of 1% (left) and 5% (right).

Figure 6. Complex phantom reconstruction using CGLS and Tikhonov as a regulariser, with noise levels of 1% (left) and 5% (right).

Both methods qualitatively. In figure 6 (using CGLS with Tikhonov), the image artefacts visually mask the four shapes which make up the original density. This may lead to threat materials or objects being misidentified (false negative errors). In addition, the artefacts introduce new “fake” densities (e.g. streaks in the top left of the image) to the original, which may be wrongly interpreted as a potential threat by security personnel (a false positive error). In figure 8 (using the iterative solver of [3]), with only a mild distortion in the image, we are less prone to such mistakes.

For a brief quantitative analysis, let the “cross” shaped object (with relative density 4) represent a detonator element and let the “triangular” density (with relative density 3) represent a small plastic explosive. Then the presence of artefacts can introduce large errors in the density estimation. For example, let us consider the left hand image in figure 6. If we take the average pixel value of the reconstructed explosive and detonator, then
Figure 7. Simple phantom reconstruction using a TV regulariser and a heuristic approach, with noise levels of 1% (left) and 5% (right).

Figure 8. Complex phantom reconstruction using a TV regulariser and a heuristic approach, with noise levels of 1% (left) and 5% (right).

The relative errors are

\[(4.8) \quad \text{errT} = 100 \times \left| \frac{\text{avgT} - 3}{3} \right| = 9.31\%, \quad \text{errC} = 100 \times \left| \frac{\text{avgC} - 4}{4} \right| = 43.9\% ,\]

where \(\text{avgT} = 2.72\) and \(\text{avgC} = 2.25\) are the average pixel values for the reconstructed plastic explosive and detonator element respectively. Let us say we were using a look up table approach to threat detection (which is a common approach). That is we look for densities (of a large enough size) in a pre–specified set of values and flag these as a potential threat. In threat detection, we cannot allow any false negatives, so if the above error rates were as expected the space of potential threats (the set of suspicious density values) would have to be increased (to allow for errors up to 44%) in order to compensate and identify the explosive, thus increasing the false positive rate.
If we now consider the same error rates for the left hand image in figure 8, then

\[(4.9)\quad \text{err}_T = 100 \times \frac{|\text{avg}_T - 3|}{3} = 0.27\%, \quad \text{err}_C = 100 \times \frac{|\text{avg}_C - 4|}{4} = 2.00\%,
\]

where in this case \(\text{avg}_T = 2.99\) and \(\text{avg}_C = 3.92\). With such a reduction in the error rate, we can safely reduce the space of potential threats (now only allowing for errors less than 2%) in our look up table and hence reduce the expected false positive rate.

5. Conclusion

Here we have introduced a new toric section transform \(\mathcal{T}\) which describes a two dimensional Compton tomography problem in airport baggage screening. A novel microlocal analysis of \(\mathcal{T}\) was presented whereby the reconstruction artefacts were explained through an analysis of the canonical relation. This was carried out by an analysis of two circle transforms \(\mathcal{T}_1\) and \(\mathcal{T}_2\), whose canonical relations \((\mathcal{C}_1\) and \(\mathcal{C}_2\)) were shown to be injective when considered separately. When we considered their disjoint union \((\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2)\), which describes the canonical relation of \(\mathcal{T}\), this was shown to be 2–1. We gave explicit expressions for the image artefacts implied by the 2–1 nature of \(\mathcal{C}\) in section 3.1.

The injectivity of \(\mathcal{T}\) was proven on the set of smooth functions \(f\) with compact support on an annulus. Here we used the parameterisation of circular arcs given by Nguyen and Truong in [14] to decompose \(\mathcal{T}f\) in terms of orthogonal special functions (exploiting the rotational symmetry of \(\mathcal{T}f\)), and then applied similar ideas to that of Cormack [1] to prove injectivity.

In section 4 we presented a practical reconstruction algorithm for the reconstruction of densities from toric section integral data using an algebraic approach. We proposed to discretize the linear operator \(\mathcal{T}\) on pixel grids (with the discrete form of \(\mathcal{T}\) stored as a sparse matrix) and to solve the corresponding set of linear equations by minimizing the least squares error with regularisation. To do this we applied the iterative techniques included in the package [3] and provided simulated reconstructions of two test phantoms (one simple and one complex) with varying levels of added pseudo random noise. Here we demonstrated the artefacts explained by our microlocal analysis through a discrete application of the normal operator of \(\mathcal{T}\) to a delta function, and showed (with a side by side comparison) that the artefacts in the reconstruction were exactly as predicted by our theory. We also showed that we could combat the artefacts in the reconstruction effectively using a heuristic iterative solver with a total variation penalty (using the code included in [3] for solving large scale image reconstruction problems), and explained how the improved artefact reduction implies a reduction in the false positive rate in the proposed application in airport baggage screening.

For further work we aim to consider more general acquisition geometries for the reconstruction of densities from toric section integral data in Compton scattering tomography. Here we have considered the particular three dimensional set of toric sections which describe the loci of scatterers for an idealised geometry for an airport baggage scanner. We wonder if the 2–1 nature of the canonical relation (or reflection artefacts) will be present for other toric section transforms and we aim to say something more concrete about this. For example, are reflection artefacts present or is the canonical relation 2–1 for any toric section transform?

References


Department of Electrical and Computer Engineering, Tufts University, Medford, MA USA

*Email address*: James.Webber@tufts.edu

Department of Mathematics, Tufts University, Medford, MA USA

*Email address*: Todd.Quinto@tufts.edu
Figure 9. Discretized toric section integrals for varying rotation angles $\alpha$ and radii $r$ are presented as images. The images are binary (the pixel value is 1 if it intersected by a toric section and 0 otherwise).
Figure 10. Predicted and observed artefacts from reconstructing a delta function close to the origin by backprojection.

Figure 11. Predicted and observed artefacts from reconstructing a delta function on the boundary of the unit ball by backprojection. The artefacts are described by a cardioid.
Figure 12. Reconstruction of simple phantom function using Landweber method and CGLS. No noise. Artefacts are present in Landweber iteration.

Figure 13. Reconstruction of complex phantom function using Landweber method and CGLS. No noise. Artefacts are present in Landweber iteration.