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MICROLOCAL ASPECTS OF COMMON OFFSET SYNTHETIC APERTURE RADAR IMAGING

Venkateswaran P. Krishnan*◇ and Eric Todd Quinto*

Department of Mathematics, Tufts University
503, Boston Avenue, Medford, MA 02155, USA

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Abstract. In this article, we analyze the microlocal properties of the linearized forward scattering operator F and the reconstruction operator F^*F appearing in bistatic synthetic aperture radar imaging. In our model, the radar source and detector travel along a line a fixed distance apart. We show that F is a Fourier integral operator, and we give the mapping properties of the projections from the canonical relation of F, showing that the right projection is a blow-down and the left projection is a fold. We then show that F^*F is a singular FIO belonging to the class $I^{3,0}$.

1. **Introduction.** In synthetic aperture radar (SAR) imaging, a region of interest on the surface of the earth is illuminated by electromagnetic waves from a moving airborne platform. The goal is to reconstruct an image of the region based on the measurement of scattered waves. For an in-depth treatment of SAR imaging, we refer the reader to [3, 2]. SAR imaging is similar to other imaging problems such as Sonar where acoustic waves are used to reconstruct the shape of objects on the ocean floor [1, 5, 20].

In monostatic SAR, the source and the receiver are located on the same moving airborne platform. In bistatic SAR, the source and the receiver are on independently moving airborne platforms. There are several advantages to considering such data acquisition geometries. The receivers, compared to the transmitters, are passive and hence are more difficult to detect. Hence, by separating their locations, the receivers alone can be in an unsafe environment, while the transmitters are in a safe environment. Furthermore, bistatic SAR systems are more resistant to electronic countermeasures such as target shaping to reduce scattering in the direction of incident waves [18].

In this paper, we consider a bistatic SAR system where the antennas have poor directivity and hence the beams do not focus on targets on the ground. We assume that the transmitter and receiver traverse a 1-dimensional curve and the backscattered data is measured at each point on this curve for a certain period of time. As in the monostatic SAR case [22], with a weak scattering assumption, the linear scattering operator that relates the unknown function that models the object on the

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◇Address as of September 2011: Tata Institute of Fundamental Research, Centre for Applicable Mathematics, Bangalore 560065, India.

ground to the data at the receiver (see [26]) is a Fourier integral operator (FIO) [17, 19, 4, 25]. Now if F is an FIO, the canonical relation Λ_F associated to F tells us how the singularities of the object are propagated to the data. The canonical relation Λ_{F^*} of the L^2 adjoint F^* of F gives us information as to how the singularities in the data are propagated back to the reconstructed object. The microlocal analysis of singularities of the object is then done by analyzing the composition $\Lambda_{F^*} \circ \Lambda_F$.

Such an analysis for monostatic SAR has been done by several authors [23, 7, 8] and is fairly well understood. In their work [23], Nolan and Cheney showed that the composition of the linearized scattering operator with its L^2 adjoint is a singular pseudodifferential operator (ΨDO) and it belongs to class of Fourier integral operators with two cleanly intersecting Lagrangians. Felea in her works [7, 8] further analyzed the properties of the composition of these operators.

In this paper, we do a similar analysis for the bistatic SAR imaging problem. Given the complications that arise in treating arbitrary transmitter and receiver trajectories, in this paper, we focus on a common offset geometry where the transmitter and receiver are at the same height above the ground, traverse the same linear trajectory at the same constant speed and spaced apart from each other by a constant distance. Furthermore, we assume that the object to be imaged is on the ground, which for simplicity, we will assume is flat. Since the measured data is two-dimensional, it is reasonable to expect that we can reconstruct a two-dimensional object.

The outline of the paper is as follows. Section 2 focuses on the preliminaries. Here we give the linearized scattering model for bistatic SAR and definitions of singularities and distributions that belong to $I^{p,l}$ classes and important results on distributions belonging to this class that are required in this paper. In Section 3, we undertake a detailed study of the canonical relation associated this FIO. This is the content of Theorem 3.6. Then we study the reconstruction operator, that is, the composition of the bistatic scattering operator with its adjoint, and show in Theorem 4.3 that this operator belongs to the class $I^{3,0}$. Our proof of 4.3 follows the ideas of [7, Theorem 1.6]. Several identities required to prove Theorem 4.3 are provided in the Appendix.

2. Preliminaries.

2.1. The bistatic linearized scattering model.

We assume that a bistatic SAR system is involved in imaging a scene. Let $\gamma_T(s)$ and $\gamma_R(s)$ for $s \in (s_0, s_1)$ be the trajectories of the transmitter and receiver respectively. The transmitter transmits electromagnetic waves that scatter off the target, which are then measured at the receiver. We are interested in obtaining a linearized model for this scattered signal.

The propagation of electromagnetic waves can be described by the scalar wave equation:

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right) E(x,t) = P(t)\delta(x - \gamma_T(s)), \tag{1}$$

where c is the speed of electromagnetic waves in the medium, $E(x,t)$ is each component of the electric field and $P(t)$ is the transmit waveform sent to the transmitter antenna located at position $\gamma_T(s)$. The wave speed c is spatially varying due to inhomogeneities present in the medium. We assume that the background in which the electromagnetic waves propagate is free space. Therefore c can be expressed as:

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2} + \widetilde{V}(x),$$

where the constant c_0 is the speed of light in free space and $\widetilde{V}(x)$ is the perturbation due to deviation from the background, which we would like to recover from backscattered waves.

Since the incident electromagnetic waves in typical radar frequencies attenuate rapidly as they penetrate the ground, we assume that $\widetilde{V}(x)$ varies only on a 2-dimensional surface. Therefore, we represent $\widetilde{V}$ as a function of the form

$$\widetilde{V}(x) = V(x)\delta_0(x_3)$$

where we assume for simplicity that the earth's surface is flat, represented by the $x = (x_1, x_2)$ plane.

The background Green's function g is then given by the solution to the following equation:

$$\left(\Delta - \frac{1}{c_0^2}\partial_t^2\right) g(x, \gamma_T(s), t) = \delta(x - \gamma_T(s))\delta(t).$$

We can explicitly write g as

$$g(x, \gamma_T(s), t) = \frac{\delta(t - |x - \gamma_T(s)|/c_0)}{4\pi|x - \gamma_T(s)|}.$$

Now the incident field E^{in} due to the source $s(x,t) = P(t)\delta(x - \gamma_T(s))$ is

$$\begin{aligned} E^{\mathrm{in}}(x,t) &= \int g(x, y, t-\tau)s(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &= \frac{P(t - |x - \gamma_T(s)|/c_0)}{4\pi|x - \gamma_T(s)|}. \end{aligned}$$

Let E denote the total field of the medium, $E = E^{\mathrm{in}} + E^{\mathrm{sc}}$. Then the scattered field can be written using the Lippman-Schwinger equation:

$$E^{\mathrm{sc}}(z,t) = \int g(z, x, t-\tau)\partial_t^2 E(x,\tau)V(x)\mathrm{d}x\mathrm{d}\tau. \tag{2}$$

We linearize (2) by the first Born approximation and write the linearized scattered wave-field at receiver location $\gamma_R(s)$:

$$\begin{aligned} E^{\mathrm{sc}}_{\mathrm{lin}}(\gamma_R(s), t) &= \int g(\gamma_R(s), x, t-\tau)\partial_t^2 E^{\mathrm{in}}(x,\tau)V(x)\mathrm{d}x\mathrm{d}\tau \\ &= \int \frac{\delta(t - \tau - |x - \gamma_R(s)|/c_0)}{4\pi|x - \gamma_R(s)|}\left(e^{-\mathrm{i}\omega(\tau - |x-\gamma_T(s)|/c_0)}\frac{\omega^2 p(\omega)}{4\pi|x - \gamma_T(s)|}\right) \\ &\quad \times V(x)\mathrm{d}\omega\mathrm{d}x\mathrm{d}\tau, \end{aligned} \tag{3}$$

where p is the Fourier transform of P.

Now, integrating (3) with respect to τ, a linearized model for the scattered signal is as follows:

$$d(s,t) := E^{\mathrm{sc}}_{\mathrm{lin}}(\gamma_R(s), t) = \int e^{-\mathrm{i}\omega(t - \frac{1}{c_0}R(s,x))}A(s,x,\omega)V(x)\mathrm{d}x\mathrm{d}\omega, \tag{4}$$

where

$$R(s,x) = |\gamma_T(s) - x| + |x - \gamma_R(s)| \tag{5}$$

and

$$A(s,x,\omega)=\omega^2p(\omega)((4\pi)^2|\gamma_T(s)-x||\gamma_R(s)-x|)^{-1}. \tag{6}$$

This function includes terms that take into account the transmitted waveform and geometric spreading factors.

We will show in Section 3 in one important case, that the transform F that maps V to (4) is a Fourier integral operator associated to a canonical relation C (Proposition 3.1), and we will prove the mapping properties of C (Propositions 3.3 and 3.5). These mapping properties tell what F does to singularities. We now define the mapping properties we need.

2.2. **Singularities and $I^{p,l}$ classes.** Here we give the definitions of the singularities associated with our operator F and its canonical relation (13), and a class of distributions required for the analysis of the composition F with its L^2 adjoint.

Definition 2.1. [12] Let M and N be manifolds of dimension n and let $f: M\to N$ be C^∞. Define $\Sigma=\{m\in M\,:\,\det(f_*)_m=0\}$.

1. f drops rank by one simply on Σ if for each $m_0\in\Sigma$, $\operatorname{rank}(f_*)_{m_0}=n-1$ and $\mathrm{d}(\det(f_*)_{m_0})\neq 0$.
2. f is a *Whitney fold* along Σ if f is a local diffeomorphism away from Σ and f drops rank by one simply on Σ, so that Σ is a smooth hypersurface and $\ker\,(f_*)_{m_0}\not\subset T_{m_0}\Sigma$ for every $m_0\in\Sigma$.
3. f is a *blow-down* along Σ if f is a local diffeomorphism away from Σ and f drops rank by one simply on Σ, so that Σ is a smooth hypersurface and $\ker(f_*)_{m_0}\subset T_{m_0}(\Sigma)$ for every $m_0\in\Sigma$.

We now define $I^{p,l}$ classes. They were first introduced by Melrose and Uhlmann, [21] Guillemin and Uhlmann [16] and Greenleaf and Uhlmann [11] and they were used in the context of radar imaging in [23, 7, 8].

Definition 2.2. Two submanifolds M and N intersect *cleanly* if $M\cap N$ is a smooth submanifold and $T(M\cap N)=TM\cap TN$.

Consider two spaces X and Y and let Λ_0 and Λ_1 and $\tilde\Lambda_0$ and $\tilde\Lambda_1$ be Lagrangian submanifolds of the product space $T^*X\times T^*Y$. If they intersect cleanly $(\tilde\Lambda_0,\tilde\Lambda_1)$ and (Λ_0,Λ_1) are equivalent in the sense that there is, microlocally, a canonical transformation χ which maps (Λ_0,Λ_1) into $(\tilde\Lambda_0,\tilde\Lambda_1)$. This leads us to the following model case.

Example 2.3. *Let $\tilde\Lambda_0=\Delta_{T^*\mathbb{R}^n}=\{(x,\xi;x,\xi)|x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n\setminus 0\}$ be the diagonal in $T^*\mathbb{R}^n\times T^*\mathbb{R}^n$ and let $\tilde\Lambda_1=\{(x',x_n,\xi',0;x',y_n,\xi',0)|x'\in\mathbb{R}^{n-1},\ \xi'\in\mathbb{R}^{n-1}\setminus 0\}$. Then, $\tilde\Lambda_0$ intersects $\tilde\Lambda_1$ cleanly in codimension 1.*

Now we define the class of product-type symbols $S^{p,l}(m,n,k)$.

Definition 2.4. $S^{p,l}(m,n,k)$ is the set of all functions $a(z,\xi,\sigma)\in C^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^k)$ such that for every $K\subset\mathbb{R}^m$ and every $\alpha\in\mathbb{Z}^m_+,\beta\in\mathbb{Z}^n_+,\gamma\in\mathbb{Z}^k_+$ there is $c_{K,\alpha,\beta}$ such that

$$|\partial_z^\alpha\partial_\xi^\beta\partial_\sigma^\gamma a(z,\xi,\sigma)|\leq c_{K,\alpha,\beta}(1+|\xi|)^{p-|\beta|}(1+|\sigma|)^{l-|\gamma|},\forall(z,\xi,\tau)\in K\times\mathbb{R}^n\times\mathbb{R}^k.$$

Since any two sets of cleanly intersecting Lagrangians are equivalent, we first define $I^{p,l}$ classes for the case in Example 2.3.

Definition 2.5. [16] Let $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_0^\infty$ and

$$u_2(x, y) = \int e^{i((x'-y'-s)\cdot\xi' + (x_n - y_n - s)\cdot\xi_n + s\cdot\sigma)} a(x, y, s; \xi, \sigma) d\xi d\sigma ds$$

with $a \in S^{p',l'}$ where $p' = p - \frac{n}{2} + \frac{1}{2}$ and $l' = l - \frac{1}{2}$.

This allows us to define the $I^{p,l}(\Lambda_0, \Lambda_1)$ class for any two cleanly intersecting Lagrangians in codimension 1 using the microlocal equivalence with the case in Example 2.3.

Definition 2.6. [16] Let $I^{p,l}(\Lambda_0, \Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2 + \sum v_i$ where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i = Aw_i$ where A is a zero order FIO associated to χ^{-1}, the canonical transformation from above, and $w_i \in I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair (Λ_0, Λ_1) to itself. By definition, $F \in I^{p,l}(\Lambda_0, \Lambda_1)$ if its Schwartz kernel belongs to $I^{p,l}(\Lambda_0, \Lambda_1)$. If $F \in I^{p,l}(\Lambda_0, \Lambda_1)$ then $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ and $F \in I^p(\Lambda_1 \setminus \Lambda_0)$ [16]. Here by $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, we mean that the Schwartz kernel of F belongs to $I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ microlocally away from Λ_1.

3. **Transmitter and receiver in a linear trajectory.** Henceforth, let us assume that the trajectory of the transmitter is

$$\gamma_T : (s_0, s_1) \to \mathbb{R}^3, \quad \gamma_T(s) = (s + \alpha, 0, h)$$

and the trajectory of the receiver is

$$\gamma_R(s) : (s_0, s_1) \to \mathbb{R}^3, \quad \gamma_R(s) = (s - \alpha, 0, h).$$

Here $\alpha > 0$ and $h > 0$ are fixed. From Equation (4), the linearized model for the data at the receiver, for $s \in (s_0, s_1)$ and $t \in (t_0, t_1)$ is

$$d(s,t) = \int e^{-\mathrm{i}\omega\left(t - \frac{1}{c_0}(|x-\gamma_T(s)| + |x - \gamma_R(s)|)\right)} A(s, x, \omega) V(x) \mathrm{d}x \mathrm{d}\omega. \tag{7}$$

We multiply $d(s,t)$ by a smooth (infinitely differentiable) function $f(s,t)$ supported in a compact subset of $(s_0, s_1) \times (t_0, t_1)$. This compensates for the discontinuities in the measurements at the end points of the rectangle $(s_0, s_1) \times (t_0, t_1)$. For simplicity, let us denote the function $f \cdot d$ as d again. We then have

$$d(s,t) = \int e^{-\mathrm{i}\omega(t - \frac{1}{c_0} R(s,x))} A(s, t, x, \omega) V(x) \mathrm{d}x \mathrm{d}\omega, \tag{8}$$

where now $A(s,t,x,\omega) = f(s,t)A(s,x,\omega)$.

Our method cannot image the point on the object that is "directly underneath" the transmitter and the receiver. This is, if the transmitter and receiver are at locations $(s + \alpha, 0, h)$ and $(s - \alpha, 0, h)$, then we cannot image the point $(s, 0, 0)$; see Remark 3.2. Therefore we modify d in Equation (8) by multiplying by another smooth function $g(s,t)$ such that

$$g \equiv 0 \quad \text{in a small neighborhood of } \left\{ \left(s, 2\frac{\sqrt{\alpha^2 + h^2}}{c_0} \right) : s_0 < s < s_1 \right\}. \tag{9}$$

For simplicity, again denote $g \cdot d$ as d and $g \cdot A$ as A. Consider,

$$FV(s,t) := d(s,t) = \int e^{-\mathrm{i}\omega(t - \frac{1}{c_0}(|x - \gamma_T(s)| + |x - \gamma_R(s)|))} A(s, t, x, \omega) V(x) \mathrm{d}x \mathrm{d}\omega. \tag{10}$$

For simplicity, let us denote the (s,t) space as Y.

We assume that the amplitude function A satisfies the following estimate: For every compact $K \in Y \times X$ and for every non-negative integer α and for every 2-indexes $\beta = (\beta_1, \beta_2)$ and γ, there is a constant C such that

$$|\partial_\omega^\alpha \partial_s^{\beta_1} \partial_t^{\beta_2} \partial_x^\gamma A(s,t,x,\omega)| \leq C(1+|\omega|)^{2-\alpha}. \tag{11}$$

This assumption is satisfied if the transmitted waveform P in (1) is approximately a Dirac delta distribution.

The phase function of the operator F,

$$\psi(s,t,x,\omega) = -\omega\left(t - \frac{1}{c_0}(|x-\gamma_T(s)| + |x-\gamma_R(s)|)\right) \tag{12}$$

is homogeneous of degree 1 in ω.

We now analyze some properties of the canonical relation of the operator F.

Proposition 3.1. *F is a Fourier integral operator of order 3/2 with canonical relation*

(13)

$$\begin{aligned}C = \Bigg\{ &\left(s, t, -\frac{\omega}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|} + \frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right), -\omega\right);\\ &\left(x_1, x_2, -\frac{\omega}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|} + \frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right), -\frac{\omega}{c_0}\left(\frac{x_2}{|x-\gamma_T(s)|} + \frac{x_2}{|x-\gamma_R(s)|}\right)\right)\\ &: c_0 t = \sqrt{(x_1-s-\alpha)^2 + x_2^2 + h^2} + \sqrt{(x_1-s+\alpha)^2 + x_2^2 + h^2}, \quad \omega \neq 0 \Bigg\}.\end{aligned}$$

Furthermore (x_1, x_2, s, ω) is a global parameterization for C.

Remark 3.2. Recall that we modified the amplitude function A to be 0 in a neighborhood of points "directly underneath the transmitter and receiver"; see (9). The exclusion of such points is required as can be seen in the definition of the canonical relation (13) above. For, if the transmitter and receiver positions are $(s+\alpha, 0, h)$ and $(s-\alpha, 0, h)$ respectively, then for $(x_1, x_2) = (s, 0)$, the cotangent vector in the canonical relation corresponding to the point $(s, 0)$ is 0. Therefore by making A to be 0 in a neighborhood of such points, we exclude a neighborhood of such points from the canonical relation in our analysis.

Proof. This is a straightforward application of the theory of FIO. Since ψ in (12) is a nondegenerate phase function with $\partial_x\psi$ and $\partial_{s,t}\psi$ nowhere zero and the amplitude A in (10) is of order 2, F is an FIO [17]. Since the amplitude is of order 2, the order of the FIO is 3/2 by [17, Definition 3.2.2]. By definition [17, Equation (3.1.2)]

$$C = \{(s, t, \partial_{s,t}\psi(x,s,t,\omega)), (x, -\partial_x\psi(x,s,t)) \,:\, \partial_\omega\psi(x,s,t,\omega) = 0\}.$$

A calculation using this definition establishes (13). Finally, it is easy to see that (x_1, x_2, s, ω) is a global parameterization of C. $\square$

In order to understand the microlocal mapping properties of F and F^*F, we consider the projections $\pi_L : T^*Y \times T^*X \to T^*Y$ and $\pi_R : T^*Y \times T^*X \to T^*X$.

Proposition 3.3. *The projection π_L restricted to C has a fold singularity on $\Sigma := \{(x_1, 0, s, \omega) : \omega \neq 0\}$.*

Proof. The projection π_L is given by

$$\pi_L(x_1,x_2,s,\omega) \tag{14}$$

$$= \left(s, \frac{1}{c_0}\left(|x-\gamma_T(s)|+|x-\gamma_R(s)|\right), -\frac{\omega}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|}+\frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right), -\omega\right)$$

We have

$$(\pi_L)_* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|}+\frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right) & \frac{1}{c_0}\left(\frac{x_2}{|x-\gamma_T(s)|}+\frac{x_2}{|x-\gamma_R(s)|}\right) & * & 0 \\ -\frac{\omega}{c_0}\left(\frac{x_2^2+h^2}{|x-\gamma_T(s)|^3}+\frac{x_2^2+h^2}{|x-\gamma_R(s)|^3}\right) & \frac{\omega}{c_0}\left(\frac{(x_1-s-\alpha)x_2}{|x-\gamma_T(s)|^3}+\frac{(x_1-s+\alpha)x_2}{|x-\gamma_R(s)|^3}\right) & * & * \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Then

$$\det(\pi_L)_* = \frac{\omega}{c_0^2}x_2\left(\frac{1}{|x-\gamma_T(s)|^2}+\frac{1}{|x-\gamma_R(s)|^2}\right)\left(1+\frac{(x_1-s)^2+x_2^2+h^2-\alpha^2}{|x-\gamma_T(s)||x-\gamma_R(s)|}\right).$$

Now Proposition 3.3 follows as a consequence of the following lemma.

Lemma 3.4. *The term*

$$1+\frac{(x_1-s)^2+x_2^2+h^2-\alpha^2}{|x-\gamma_T(s)||x-\gamma_R(s)|}.$$

is positive for all $x\in\mathbb{R}^2$, $s\in\mathbb{R}$ and h and α positive.

Proof. This is a straightforward calculation that is made simpler if one lets $S = x_1-s$, $T=\sqrt{x_2^2+h^2}$ and then puts the term over a common denominator. Finally, one shows that the numerator is positive by isolating the square roots (absolute values), squaring, and simplifying to infer that, since $4T^2\alpha^2>0$, the lemma is true. ☐

Now returning to the proof of the Proposition 3.3, we have that $\det(\pi_L)_*=0$ if and only if $x_2=0$. Hence $\det(\pi_L)_*$ vanishes on the set Σ and Lemma 3.4 again shows that $\mathrm{d}(\det(\pi_L)_*)$ on Σ is non-vanishing. This implies that π_L drops rank by one simply on Σ. Alternately, one can also see that $(\pi_L)_*|_\Sigma$ has rank 3 by letting $x_2=0$ in (15). Furthermore, $(\pi_L)_*$ has full rank except on Σ, because $\det(\pi_L)_*$ is non-vanishing except on Σ.

Now it remains to show that $T\Sigma\cap\mathrm{Kernel}(\pi_L)_*=\{0\}$. But this follows from the fact that $\mathrm{Kernel}(\pi_L)_*=\mathrm{span}(\frac{\partial}{\partial x_2})$, but $T\Sigma=\mathrm{span}(\frac{\partial}{\partial x_1},\frac{\partial}{\partial s},\frac{\partial}{\partial \omega})$. This concludes the proof of Proposition 3.3. ☐

Proposition 3.5. *Consider the projection $\pi_R: T^*Y\times T^*X\to T^*X$. The restriction of the projection to C has a blowdown singularity on Σ.*

Proof. We have

$$\pi_R(x_1,x_2,s,\omega) = \tag{16}$$

$$= \left(x_1,x_2,-\frac{\omega}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|}+\frac{x_1-s-\alpha}{|x-\gamma_R(s)|}\right), -\frac{\omega}{c_0}\left(\frac{x_2}{|x-\gamma_T(s)|}+\frac{x_2}{|x-\gamma_R(s)|}\right)\right).$$

Now

$$(\pi_R)_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & -\frac{\omega}{c_0}\left(\frac{x_2^2+h^2}{|x-\gamma_T(s)|^3} + \frac{x_2^2+h^2}{|x-\gamma_R(s)|^3}\right) & -\frac{1}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|} + \frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right) \\ * & * & \frac{\omega}{c_0}\left(\frac{(x_1-s-\alpha)x_2}{|x-\gamma_T(s)|^3} + \frac{(x_1-s+\alpha)x_2}{|x-\gamma_R(s)|^3}\right) & -\frac{1}{c_0}\left(\frac{x_2}{|x-\gamma_T(s)|} + \frac{x_2}{|x-\gamma_R(s)|}\right) \end{pmatrix}$$

From this we see that $\mathrm{Kernel}(\pi_R)_* \subset T\Sigma$. Since $\det((\pi_R)_*) = \det((\pi_L)_*)$, π_R drops rank by one simply along Σ. Therefore the projection π_R has a blowdown singularity along Σ. $\square$

We summarize what we have proved in this section by the following theorem:

Theorem 3.6. *The operator F defined in (10) is a Fourier integral operator of order $3/2$. The canonical relation C associated to F defined in (13) satisfies the following: The projections π_L and π_R defined in (14) and (16) are a fold and blowdown respectively.*

4. Image reconstruction.

Next, we study the composition of F with F^*. This composition is given as follows:

$$F^*FV(x) = \int e^{\mathrm{i}\left(\omega(t-\frac{1}{c_0}(|x-\gamma_R(s)|+|x-\gamma_R(s)|))-\widetilde{\omega}(t-\frac{1}{c_0}(|y-\gamma_T(s)|+|y-\gamma_R(s)|))\right)} \times \overline{A(x,s,t,\omega)}A(y,s,t,\widetilde{\omega})V(y)\mathrm{d}s\mathrm{d}t\mathrm{d}\omega\mathrm{d}\widetilde{\omega}\mathrm{d}y$$

After an application of the method of stationary phase, we can write the kernel of the operator F^*F as

$$K(x,y) = \int e^{\mathrm{i}\frac{\omega}{c_0}(|y-\gamma_T(s)|+|y-\gamma_R(s)|-(|x-\gamma_T(s)|+|x-\gamma_R(s)|))}\widetilde{A}(x,y,s,\omega)\mathrm{d}s\mathrm{d}\omega.$$

Therefore the phase function of the kernel $K(x,y)$ is

$$(17)\quad \phi(x,y,s,\omega) = \frac{\omega}{c_0}\left(|y-\gamma_T(s)| + |y-\gamma_R(s)| - (|x-\gamma_T(s)| + |x-\gamma_R(s)|)\right).$$

Proposition 4.1.

$$WF(K)' \subset \Delta \cup \Lambda$$

where $\Delta := \{(x_1,x_2,\xi_1,\xi_2;x_1,x_2,\xi_1,\xi_2)\}$ and $\Lambda := \{(x_1,x_2,\xi_1,\xi_2;x_1,-x_2,\xi_1,-\xi_2)\}$. Here for a point $x=(x_1,x_2)$, the covectors (ξ_1,ξ_2) are non-zero multiples of the vector $(-\partial_{x_1}R(s,x),-\partial_{x_2}R(s,x))$, where R is defined in (5).

Proof. Using the Hörmander-Sato Lemma, we have

$$WF(K)' \subset$$

$$\Bigg\{\left(x_1,x_2,-\frac{\omega}{c_0}\left(\frac{x_1-s-\alpha}{|x-\gamma_T(s)|}+\frac{x_1-s+\alpha}{|x-\gamma_R(s)|}\right),-\frac{\omega}{c_0}\left(\frac{x_2}{|x-\gamma_T(s)|}+\frac{x_2}{|x-\gamma_R(s)|}\right)\right);$$
$$\left(y_1,y_2,-\frac{\omega}{c_0}\left(\frac{y_1-s-\alpha}{|y-\gamma_T(s)|}+\frac{y_1-s+\alpha}{|y-\gamma_R(s)|}\right),-\frac{\omega}{c_0}\left(\frac{y_2}{|y-\gamma_T(s)|}+\frac{y_2}{|y-\gamma_R(s)|}\right)\right):$$
$$|x-\gamma_T(s)|+|x-\gamma_R(s)| = |y-\gamma_T(s)|+|y-\gamma_R(s)|,$$
$$\frac{x_1-s-\alpha}{|x-\gamma_T(s)|}+\frac{x_1-s+\alpha}{|x-\gamma_R(s)|} = \frac{y_1-s-\alpha}{|y-\gamma_T(s)|}+\frac{y_1-s+\alpha}{|y-\gamma_R(s)|},\quad \omega\neq 0\Bigg\}.$$

We now obtain a relation between (x_1, x_2) and (y_1, y_2). This is given by the following lemma.

Lemma 4.2. *For all s, the set of all (x_1, x_2), (y_1, y_2) that satisfy*

$$|x - \gamma_T(s)| + |x - \gamma_R(s)| = |y - \gamma_T(s)| + |y - \gamma_R(s)|, \tag{18}$$

$$\frac{x_1 - s - \alpha}{|x - \gamma_T(s)|} + \frac{x_1 - s + \alpha}{|x - \gamma_R(s)|} = \frac{y_1 - s - \alpha}{|y - \gamma_T(s)|} + \frac{y_1 - s + \alpha}{|y - \gamma_R(s)|}. \tag{19}$$

necessarily satisfy the following relations: $x_1 = y_1$ and $x_2 = \pm y_2$.

Proof. In order to show this, we will consider (18) and (19) as functions of $\mathbb{R}^3$ by replacing h in these expressions with $x_3 - h$. We then transform these expressions using the coordinates (20) and then set $x_3 = y_3 = 0$ to prove the lemma.

Consider the following change of coordinates, the so called prolate spheroidal coordinates:

$$\begin{array}{ll} x_1 = s + \alpha \cosh\rho \cos\theta & y_1 = s + \alpha \cosh\rho' \cos\theta' \\ x_2 = \alpha \sinh\rho \sin\theta \cos\varphi & y_2 = \alpha \sinh\rho' \sin\theta' \cos\varphi' \\ x_3 = h + \alpha \sinh\rho \sin\theta \sin\varphi & y_3 = h + \alpha \sinh\rho' \sin\theta' \sin\varphi' \end{array} \tag{20}$$

where s, $\alpha > 0$ and $h > 0$ are fixed and $\rho \in [0, \infty)$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. This a well-defined coordinate system except for $\rho = 0$ and $\theta = 0, \pi$.

This coordinate system has also been used in the context of radar imaging by T. Dowling in his thesis [6].

In the coordinate system (20), we have

$$\begin{array}{ll} |x - \gamma_T(s)| = \alpha(\cosh\rho - \cos\theta), & |x - \gamma_R(s)| = \alpha(\cosh\rho + \cos\theta), \\ \frac{x_1 - s - \alpha}{|x - \gamma_T(s)|} = \frac{\cosh\rho\cos\theta - 1}{\cosh\rho - \cos\theta}, & \frac{x_1 - s + \alpha}{|x - \gamma_R(s)|} = \frac{\cosh\rho\cos\theta + 1}{\cosh\rho + \cos\theta}. \end{array} \tag{21}$$

The terms involving y are obtained similarly. Now (18) and (19) transform as follows:

$$2\cosh\rho = 2\cosh\rho'$$

$$\frac{\cosh\rho\cos\theta - 1}{\cosh\rho - \cos\theta} + \frac{\cosh\rho\cos\theta + 1}{\cosh\rho + \cos\theta} = \frac{\cosh\rho'\cos\theta' - 1}{\cosh\rho' - \cos\theta'} + \frac{\cosh\rho'\cos\theta' + 1}{\cosh\rho' + \cos\theta'}.$$

Using the first equality in the second equation, we have

$$\frac{\cos\theta}{\cosh^2\rho - \cos^2\theta} = \frac{\cos\theta'}{\cosh^2\rho - \cos^2\theta'}.$$

This gives $\cos\theta = \cos\theta'$. Therefore $\theta = 2n\pi \pm \theta'$, which then gives $\sin\theta = \pm\sin\theta'$. Therefore, in terms of (x_1, x_2) and (y_1, y_2), we have $x_1 = y_1$ and $x_2 = \pm y_2$. □

Now to finish the proof of Proposition 4.1, when $x_1 = y_1$ and $x_2 = y_2$, there is contribution to $WF(K)'$ contained in the diagonal set $\Delta := \{(x_1, x_2, \xi_1, \xi_2; x_1, x_2, \xi_1, \xi_2)\}$ and when $x_1 = y_1$ and $x_2 = -y_2$, we have a contribution to $WF(K)'$ contained in Λ, where $\Lambda := \{(x_1, x_2, \xi_1, \xi_2; x_1, -x_2, \xi_1, -\xi_2)\}$. □

Note that the Lagrangians Δ and Λ intersect cleanly. For, we have,

$$\Delta \cap \Lambda = \{(x_1, 0, \xi_1, 0; x_1, 0, \xi_1, 0)\}$$

which is a submanifold of codimension 2 in both Δ and Λ. Also for any $p \in \Delta \cap \Lambda$, we have

$$T_p\Delta = \{(v_1, v_2, w_1, w_2; v_1, v_2, w_1, w_2) : v_1, v_2, w_1, w_2 \in \mathbb{R}\}$$

and

$$T_p\Lambda = \{(v_1, v_2, w_1, w_2; v_1, -v_2, w_1, -w_2) : v_1, v_2, w_1, w_2 \in \mathbb{R}\}.$$

Also

$$T_p(\Delta \cap \Lambda) = \{(v_1, 0, w_1, 0; v_1, 0, w_1, 0) : v_1, w_1 \in \mathbb{R}\}$$

which is the same as $T_p\Delta \cap T_p\Lambda$, showing that the Lagrangians Δ and Λ intersect cleanly. Therefore the clean intersection calculus applies to the composition of F and F^*, and $F^*F \in I^{p,l}(\Delta, \Lambda)$ for some p and l.

We use the following convention in the theorem below. The cotangent variables corresponding to x and y are denoted as ξ and η respectively. Then note that $\xi = \partial_x\phi$ and $\eta = -\partial_y\phi$.

Theorem 4.3. *Let F be given by (10). Then $F^*F \in I^{3,0}(\Delta, \Lambda)$.*

Proof. We follow the proof of [7, Theorem 1.6] closely to prove this result. Recall that Δ and Λ are defined by

$$\Delta = \{x_1 - y_1 = x_2 - y_2 = \xi_1 - \eta_1 = \xi_2 - \eta_2 = 0\},$$
$$\Lambda = \{x_1 - y_1 = x_2 + y_2 = \xi_1 - \eta_1 = \xi_2 + \eta_2 = 0\}.$$

The ideal of functions that vanish on $\Delta \cup \Lambda$ is generated by

$$\widetilde{p}_1 = x_1 - y_1, \quad \widetilde{p}_2 = x_2^2 - y_2^2, \quad \widetilde{p}_3 = \xi_1 - \eta_1, \quad \widetilde{p}_4 = (x_2 + y_2)(\xi_2 - \eta_2),$$
$$\widetilde{p}_5 = (x_2 - y_2)(\xi_2 + \eta_2), \quad \widetilde{p}_6 = \xi_2^2 - \eta_2^2.$$

Let $p_i = q_i\widetilde{p}_i$, for $1 \le i \le 6$, where q_1, q_2 are homogeneous of degree 1 in (ξ, η), q_3, q_4 and q_5 are homogeneous of degree 0 in (ξ, η) and q_6 is homogeneous of degree -1 in (ξ, η). Let P_i be pseudodifferential operators with principal symbols p_i for $1 \le i \le 6$.

In order to prove that $F^*F \in I^{p,l}(\Delta, \Lambda)$, for some p, l, we have to show that $P_iK \in H^{s_0}_{\mathrm{loc}}$ for some s_0, for $1 \le i \le 6$. By [4, Proposition 4.3.1], we have the following (up to lower order terms):

$$P_iK(x, y) = \int e^{\mathrm{i}\frac{\omega}{c_0}(R(s,y)-R(s,x))}\widetilde{A}(x, y, s, \omega)p_i(x, y, -\partial_x R(s, x), -\partial_y R(s, y))\mathrm{d}s\mathrm{d}\omega$$

We show in the Appendix that each $\widetilde{p}_i$ can be expressed in the following forms:

$$\widetilde{p}_1 = \frac{f_{11}(x, y, s)}{\omega}\partial_s\phi + f_{12}(x, y, s)\partial_\omega\phi \tag{22}$$

$$\widetilde{p}_2 = \frac{f_{21}(x, y, s)}{\omega}\partial_s\phi + f_{22}(x, y, s)\partial_\omega\phi \tag{23}$$

$$\widetilde{p}_3 = -\partial_s\phi \tag{24}$$

$$\widetilde{p}_4 = f_{41}(x, y, s)\partial_s\phi + \omega f_{42}(x, y, s)\partial_\omega\phi \tag{25}$$

$$\widetilde{p}_5 = f_{51}(x, y, s)\partial_s\phi + \omega f_{52}(x, y, s)\partial_\omega\phi \tag{26}$$

$$\widetilde{p}_6 = \omega f_{61}(x, y, s)\partial_s\phi + \omega^2 f_{62}(x, y, s)\partial_\omega\phi. \tag{27}$$

Here f_{ij} for $1 \le i \le 6$ and $j = 1, 2$ are smooth functions.

Therefore

$$\begin{aligned}P_1K(x,y) &= \int e^{\mathrm{i}\phi(x,y,s,\omega)}\widetilde{A}(x,y,s,\omega)q_1\left(\frac{f_{11}(x,y,s)}{\omega}\partial_s\phi + f_{12}(x,y,s)\partial_\omega\phi\right)\mathrm{d}s\mathrm{d}\omega\\ &= \int \partial_s\left(e^{\mathrm{i}\phi(x,y,s,\omega)}\right)\frac{q_1}{\mathrm{i}\omega}\widetilde{A}(x,y,s,\omega)f_{11}(x,y,s)\mathrm{d}s\mathrm{d}\omega\\ &+ \int \partial_\omega\left(e^{\mathrm{i}\phi(x,y,s,\omega)}\right)\frac{q_1}{\mathrm{i}}\widetilde{A}(x,y,s,\omega)f_{12}(x,y,s)\mathrm{d}s\mathrm{d}\omega\end{aligned}$$

By integration by parts

$$\begin{aligned}&= -\Bigg\{\int e^{\mathrm{i}\phi(x,y,s,\omega)}\partial_s\left(\frac{q_1}{\mathrm{i}\omega}\widetilde{A}(x,y,s,\omega)f_{11}(x,y,s)\right)\mathrm{d}s\mathrm{d}\omega\\ &+ \int e^{\mathrm{i}\phi(x,y,s,\omega)}\partial_\omega\left(\frac{q_1}{\mathrm{i}}\widetilde{A}(x,y,s,\omega)f_{12}(x,y,s)\right)\mathrm{d}s\mathrm{d}\omega\Bigg\}.\end{aligned}$$

Note that q_1 is homogeneous of degree 1 in ω, and $\widetilde{A}$ is a symbol of order 4, hence each amplitude term in the sum above is of order 4.

Therefore by [4, Theorem 2.2.1], we have that $P_1K \in H^{s_0}_{\mathrm{loc}}$ for some s_0.

A similar argument works for each of the other five pseudodifferential operators. Hence by [11, Proposition 1.35], we have that $F^*F \in I^{p,l}(\Delta,\Lambda)$. Because C is a local canonical graph away from Σ, the transverse intersection calculus applies for the composition F^*F away from Σ. Hence F^*F is of order 3 on $\Delta\setminus\Sigma$ and $\Lambda\setminus\Sigma$. Since F^*F is of order $p+l$ on $\Delta\setminus\Sigma$ and is of order p on $\Lambda\setminus\Sigma$, we have that $p=3$ and $l=0$. Therefore the theorem is proved. □

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Appendix A. Here we prove the identities (22) through (27) that are required in the proof of Theorem 4.3. For convenience, and without loss of generality, we will assume $c_0 = 1$.

In obtaining these identities, it is easier to work in the coordinate system defined in (20). We will work with the extension $\widetilde{\phi}$ of the phase function ϕ to $\mathbb{R}^3$ defined by

$$\begin{aligned}\widetilde{\phi} = \frac{\omega}{c_0}\Bigg(&\sqrt{(y_1-s-\alpha)^2+y_2^2+(y_3-h)^2}+\sqrt{(y_1-s+\alpha)+y_2^2+(y_3-h)^2}\\ &-\left(\sqrt{(x_1-s-\alpha)^2+x_2^2+(x_3-h)^2}+\sqrt{(x_1-s+\alpha)+x_2^2+(x_3-h)^2}\right)\Bigg).\end{aligned}$$

Then, using the facts that

$$\partial_\omega\widetilde{\phi}|_{x_3=y_3=0} = \partial_\omega\phi \quad \text{and} \quad \partial_s\widetilde{\phi}|_{x_3=y_3=0} = \partial_s\phi, \tag{28}$$

we set the third coordinate $x_3 = y_3 = 0$ to obtain the required identities.

A.1. **Expression for $x_1 - y_1$.** We now obtain an expression for $x_1 - y_1$ of the form

$$x_1 - y_1 = \frac{f_{11}(x,y,s)}{\omega}\partial_s\phi + f_{12}(x,y,s)\partial_\omega\phi, \tag{29}$$

where f_{11} and f_{12} are smooth functions.

That is, denoting $A_1 = x_1 - y_1$, we would like to obtain an expression of the form (29) involving $\partial_s\phi$ and $\partial_\omega\phi$ for

$$A_1 = \alpha(\cosh\rho\cos\theta - \cosh\rho'\cos\theta'). \tag{30}$$

We have

$$\partial_\omega\widetilde{\phi} = 2\alpha(\cosh\rho' - \cosh\rho)$$

and

$$\begin{aligned}\partial_s\widetilde{\phi} &= -\,\omega\Bigg(\left(\frac{\cosh\rho'\cos\theta' - 1}{\cosh\rho' - \cos\theta'} + \frac{\cosh\rho'\cos\theta' + 1}{\cosh\rho' + \cos\theta'}\right)\\ &\qquad - \left(\frac{\cosh\rho\cos\theta - 1}{\cosh\rho - \cos\theta} + \frac{\cosh\rho\cos\theta + 1}{\cosh\rho + \cos\theta}\right)\Bigg)\\ &= 2\omega\left(\frac{\sinh^2\rho\cos\theta}{\cosh^2\rho - \cos^2\theta} - \frac{\sinh^2\rho'\cos\theta'}{\cosh^2\rho' - \cos^2\theta'}\right) \end{aligned}\tag{31}$$

using $\cosh^2\rho\cos\theta - \cos\theta = \sinh^2\rho\cos\theta$. Now

$$\cos\theta\frac{\partial_\omega\widetilde{\phi}}{2} = \alpha\left(\cosh\rho'\cos\theta' - \cosh\rho\cos\theta\right) + \alpha\cosh\rho'\left(\cos\theta - \cos\theta'\right). \tag{32}$$

Then

$$A_1 = -\frac{\cos\theta}{2}\partial_\omega\widetilde{\phi} + \alpha\cosh\rho'(\cos\theta - \cos\theta'). \tag{33}$$

Adding and subtracting $\frac{\sinh^2\rho\cos\theta'}{\cosh^2\rho - \cos^2\theta'}$ inside (31), we have

$$\begin{aligned}\partial_s\widetilde{\phi} = 2\omega\Bigg(&\underbrace{\frac{\sinh^2\rho\cos\theta}{\cosh^2\rho - \cos^2\theta} - \frac{\sinh^2\rho\cos\theta'}{\cosh^2\rho - \cos^2\theta'}}_{\mathrm{I}}\\ &+ \underbrace{\frac{\sinh^2\rho\cos\theta'}{\cosh^2\rho - \cos^2\theta'} - \frac{\sinh^2\rho'\cos\theta'}{\cosh^2\rho' - \cos^2\theta'}}_{\mathrm{II}}\Bigg).\end{aligned}$$

Simplifying I and II, we have,

$$\mathrm{I} = \frac{(\cos\theta - \cos\theta')(\sinh^2\rho)(\cosh^2\rho + \cos\theta\cos\theta')}{(\cosh^2\rho - \cos^2\theta)(\cosh^2\rho - \cos^2\theta')}$$

and

$$\begin{aligned}\mathrm{II} &= \frac{\cos\theta'\sin^2\theta'(\cosh\rho - \cosh\rho')(\cosh\rho + \cosh\rho')}{(\cosh^2\rho' - \cos^2\theta')(\cosh^2\rho - \cos^2\theta')}\\ &= -\frac{\cos\theta'\sin^2\theta'(\cosh\rho + \cosh\rho')}{(\cosh^2\rho' - \cos^2\theta')(\cosh^2\rho - \cos^2\theta')}\cdot\frac{\partial_\omega\widetilde{\phi}}{2\alpha}\end{aligned}$$

where we have used the fact $\sinh^2\rho - \sinh^2\rho' = \cosh^2\rho - \cosh^2\rho'$.

Using these calculations, we see

$$\cos\theta - \cos\theta' = \left(\frac{(\cosh^2\rho - \cos^2\theta)(\cosh^2\rho - \cos^2\theta')}{(\cosh^2\rho - 1)(\cosh^2\rho + \cos\theta\cos\theta')}\right) \times \left(\frac{\partial_s\widetilde{\phi}}{2\omega} + \frac{\cos\theta'\sin^2\theta'(\cosh\rho + \cosh\rho')}{(\cosh^2\rho' - \cos^2\theta')(\cosh^2\rho - \cos^2\theta')}\cdot\frac{\partial_\omega\widetilde{\phi}}{2\alpha}\right). \tag{34}$$

Now setting $x_3 = y_3 = 0$ and using (33) , we have,

$$\begin{aligned} A_1 =& \frac{\alpha\cosh\rho'(\cosh^2\rho - \cos^2\theta)(\cosh^2\rho - \cos^2\theta')}{(\cosh^2\rho - 1)(\cosh^2\rho + \cos\theta\cos\theta')}\frac{\partial_s\phi}{2\omega} \\ &- \frac{1}{2}\left(\cos\theta - \frac{\cosh\rho'\cos\theta'\sin^2\theta'(\cosh\rho + \cosh\rho')}{(\cosh^2\rho - 1)(\cosh^2\rho + \cos\theta\cos\theta')}\cdot\frac{(\cosh^2\rho - \cos^2\theta)}{(\cosh^2\rho' - \cos^2\theta')}\right)\partial_\omega\phi. \end{aligned}$$

We can see that no denominator in this expression is zero for $x_3 = 0$ (since $\cosh\rho > 1 \geq \cos\theta$ if $x_3 = 0$) and so this expression for A_1 is defined and smooth for all values of the coordinates.

We can write A_1 in the Cartesian coordinate system as follows. First, for simplicity, let

$$X_1 = |x - \gamma_T(s)| = \sqrt{(x_1 - s - \alpha)^2 + x_2^2 + h^2}, \tag{35}$$

$$X_2 = |x - \gamma_R(s)| = \sqrt{(x_1 - s + \alpha)^2 + x_2^2 + h^2}, \tag{36}$$

$$Y_1 = |y - \gamma_T(s)| = \sqrt{(y_1 - s - \alpha)^2 + y_2^2 + h^2}, \tag{37}$$

$$Y_2 = |y - \gamma_R(s)| = \sqrt{(y_1 - s + \alpha)^2 + y_2^2 + h^2}. \tag{38}$$

Then using these expressions and (21) we see that

$$\begin{aligned} A_1 =& \frac{\alpha\left(\frac{Y_1+Y_2}{2\alpha}\right)\left(\frac{X_1X_2}{\alpha^2}\right)\left(\frac{(X_1+X_2)^2}{4\alpha^2} - \frac{4(y_1-s)^2}{(Y_1+Y_2)^2}\right)}{2\omega\left(\frac{(X_1+X_2)^2}{4\alpha^2} - 1\right)\left(\frac{(X_1+X_2)^2}{4\alpha^2} + \frac{4(x_1-s)(y_1-s)}{(X_1+X_2)(Y_1+Y_2)}\right)}\partial_s\phi \\ &- \frac{1}{2}\left(\frac{2(x_1-s)}{X_1+X_2} - \frac{X_1X_2\left(\frac{2(y_1-s)}{Y_1+Y_2}\right)\left(1 - \frac{4(y_1-s)^2}{(Y_1+Y_2)^2}\right)\left(\frac{X_1+X_2+Y_1+Y_2}{2\alpha}\right)}{Y_1Y_2\left(\frac{(X_1+X_2)^2}{4\alpha^2} - 1\right)\left(\frac{(X_1+X_2)^2}{4\alpha^2} + \frac{4(x_1-s)(y_1-s)}{(X_1+X_2)(Y_1+Y_2)}\right)}\right)\partial_\omega\phi. \end{aligned}$$

We see from this second expression for A_1 that the coefficient functions are smooth in (x, y, s) since the expressions (35)-(38) are non-zero and smooth.

A.2. Expression for $x_2^2 - y_2^2$.

Now we write $x_2^2 - y_2^2$ in the form

$$A_2 := x_2^2 - y_2^2 = \frac{f_{21}(x,y,s)}{\omega}\partial_s\phi + f_{22}(x,y,s)\partial_\omega\phi, \tag{39}$$

where f_{21} and f_{22} are smooth functions. A_2 in the coordinate system (20) is

$$A_2 := \alpha^2(\sinh^2\rho\sin^2\theta\cos^2\varphi - \sinh^2\rho'\sin^2\theta'\cos^2\varphi')$$

$$= \alpha^2(\sinh^2\rho\sin^2\theta - \sinh^2\rho'\sin^2\theta') \tag{40}$$

$$+ \alpha^2(\sinh^2\rho'\sin^2\theta'\sin^2\varphi' - \sinh^2\rho\sin^2\theta\sin^2\varphi). \tag{41}$$

For $x_3 = y_3 = 0$, (41) is 0. So it is enough to obtain an expression of the form (39) for (40), which we still denote by A_2.

Using the following identities,

$$\sinh^2\rho = \cosh^2\rho - 1 \text{ and } \sin^2\theta = 1-\cos^2\theta,$$

we have,

$$\begin{aligned}
A_2 &= \alpha^2(\sinh^2\rho\sin^2\theta - \sinh^2\rho'\sin^2\theta')\\
&= \alpha^2\left((\cosh^2\rho-\cosh^2\rho') + (\cos^2\theta-\cos^2\theta') - (\cosh^2\rho\cos^2\theta - \cosh^2\rho'\cos^2\theta')\right)\\
&= -\frac{\alpha}{2}(\cosh\rho+\cosh\rho')\partial_\omega\widetilde{\phi} + \alpha^2(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')\\
&- \alpha(\cosh\rho\cos\theta+\cosh\rho'\cos\theta')A_1\\
&= -\frac{\alpha}{2}(\cosh\rho+\cosh\rho')\partial_\omega\widetilde{\phi} + \alpha^2(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')\\
&- \alpha(\cosh\rho\cos\theta+\cosh\rho'\cos\theta')(-\frac{\cos\theta}{2}\partial_\omega\phi + \alpha\cosh\rho'(\cos\theta-\cos\theta'))\\
&= \frac{\alpha}{2}\Big(-(\cosh\rho+\cosh\rho') + (\cosh\rho\cos\theta+\cosh\rho'\cos\theta')\cos\theta\Big)\partial_\omega\widetilde{\phi}\\
&+ \alpha^2\Big((\cos\theta+\cos\theta') - (\cosh\rho\cos\theta+\cosh\rho'\cos\theta')\cosh\rho'\Big)(\cos\theta-\cos\theta').
\end{aligned}$$

Now we use the expression for $\cos\theta - \cos\theta'$ in Equation (34) and set $x_3 = y_3 = 0$; this shows that $x_2^2 - y_2^2$ can be written in the form

$$A_2 = \frac{f_{21}(x,y,s)}{\omega}\partial_s\phi + f_{22}(x,y,s)\partial_\omega\phi.$$

A.3. **Expression for $\xi_1 - \eta_1$.** Now we consider $\xi_1 - \eta_1$, where we recall that $(\xi_1,\xi_2) = \partial_x\phi$ and $(\eta_1,\eta_2) = -\partial_y\phi$ are the cotangent variables corresponding to (x_1,x_2) and (y_1,y_2) respectively.

Then note that $\xi_1 - \eta_1$ is $\partial_{x_1}\phi + \partial_{y_1}\phi$. But this is the same as $-\partial_s\phi$. Hence

$$A_3 := \xi_1 - \eta_1 = -\partial_s\phi.$$

A.4. **Expression for $(x_2-y_2)(\xi_2+\eta_2)$.** We have (up to a negative sign)

$$\begin{aligned}
(x_2-y_2)(\xi_2+\eta_2) &= \omega(x_2-y_2)\left(\frac{x_2}{|x-\gamma_T|} + \frac{x_2}{|x-\gamma_R|} + \frac{y_2}{|y-\gamma_T|} + \frac{y_2}{|y-\gamma_R|}\right)\\
&= 2\omega\Bigg(\frac{x_2^2\cosh\rho}{\cosh^2\rho-\cos^2\theta} - \frac{y^2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\\
&\qquad + \frac{x_2y_2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'} - \frac{x_2y_2\cosh\rho}{\cosh^2\rho-\cos^2\theta}\Bigg)\\
&= 2\omega\Bigg(\frac{x_2^2\cosh\rho}{\cosh^2\rho-\cos^2\theta} - \frac{x_2^2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\\
&+ (x_2^2-y_2^2)\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\\
&+ \frac{x_2y_2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'} - \frac{x_2y_2\cosh\rho}{\cosh^2\rho-\cos^2\theta}\Bigg),
\end{aligned}$$

Here we have added and subtracted $\frac{x_2^2 \cosh\rho'}{\cosh^2\rho' - \cos^2\theta'}$ in the previous equation. Simplifying this we get,

$$\begin{aligned}(x_2-y_2)(\xi_2+\eta_2) =& 2\omega\Bigg((x_2^2-x_2y_2)\Bigg[\frac{(\cosh\rho\cosh\rho'+\cos^2\theta)(\cosh\rho'-\cosh\rho)}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\\ &+\frac{\cosh\rho(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\Bigg]\\ &+(x_2^2-y_2^2)\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\Bigg).\end{aligned}$$

Now note that $\cosh\rho' - \cosh\rho = \frac{\partial_\omega\phi}{2\alpha}$ and we already have expressions for $\cos\theta - \cos\theta'$ and $x_2^2 - y_2^2$ involving combinations of $\partial_\omega\phi$ and $\partial_s\phi$.

Hence we can write $(x_2-y_2)(\xi_2+\eta_2)$ in the form

$$(x_2-y_2)(\xi_2+\eta_2) = f_{41}(x,y,s)\partial_s\phi + \omega f_{42}(x,y,s)\partial_\omega\phi.$$

For future reference, note that our calculation in this section shows that

(42)
$$\begin{aligned}\frac{\cosh\rho}{\cosh^2\rho-\cos^2\theta}-\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'} =& \frac{(\cosh\rho\cosh\rho'+\cos^2\theta)(\cosh\rho'-\cosh\rho)}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\\ &+\frac{\cosh\rho(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\end{aligned}$$

A.5. **Expression for** $(x_2+y_2)(\xi_2-\eta_2)$**.** This is very similar to the derivation of the expression we obtained for $(x_2-y_2)(\xi_2+\eta_2)$.

A.6. **Expression for** $\rho_2^2-\eta_2^2$**.** Using (36) and (38), we have

$$\begin{aligned}\xi_2^2-\eta_2^2 &= \omega^2\left(\left(\frac{x_2}{|x-\gamma_T|}+\frac{x_2}{|x-\gamma_R|}\right)^2-\left(\frac{y_2}{|y-\gamma_T|}+\frac{y_2}{|y-\gamma_R|}\right)^2\right)\\ &= 4\omega^2\left(x_2^2\frac{\cosh^2\rho}{(\cosh^2\rho-\cos^2\theta)^2}-y_2^2\frac{\cosh^2\rho'}{(\cosh^2\rho'-\cos^2\theta')^2}\right)\\ &= 4\omega^2\Bigg\{x_2^2\left(\frac{\cosh^2\rho}{(\cosh^2\rho-\cos^2\theta)^2}-\frac{\cosh^2\rho'}{(\cosh^2\rho'-\cos^2\theta')^2}\right)\\ &\quad+(x_2^2-y_2^2)\frac{\cosh^2\rho'}{(\cosh^2\rho'-\cos^2\theta')^2}\Bigg\}.\end{aligned}$$

Now using the computations for $x_2^2-y_2^2$ and $(x_2-y_2)(\rho_2+\eta_2)$, in particular (42), we can write $\xi_2^2-\eta_2^2$ in the form

$$\xi_2^2-\eta_2^2 = \omega f_{61}(x,y,s)\partial_s\phi + \omega^2 f_{62}(x,y,s)\partial_\omega\phi$$

for smooth functions f_{61}, f_{62}.

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E-mail address: venkyp.krishnan@gmail.com
E-mail address: todd.quinto@tufts.edu