GEOMETRIC GROUP THEORY NOTES

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1. September 7, 2010

Definition. Let Γ be a group. Take $A \subset \Gamma$ to be any subset. The *subgroup generated by* A, denoted $\langle A \rangle$, is the smallest subgroup of Γ containing A. Equivalently, $\langle A \rangle$ is the intersection of all subgroups of Γ containing A. Equivalently,

$$\langle A \rangle = \{a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \mid a_i \in A, \epsilon_i = \pm 1\},$$

the set of finite products of elements of $A \cup A^{-1}$.

Notation. We will frequently use Γ to denote a group. If A is a set, $A^{-1} = \{a^{-1} \mid a \in A\}$.

Remark. $\langle A \rangle$ is characterized by the following properties:

(1) $A \subset \langle A \rangle$
(2) $\langle A \rangle \leq \Gamma$
(3) If $G \leq \Gamma$ and $A \subset G$, then $\langle A \rangle \subset G$.

Definition. If $A \subset \Gamma$ and $\langle A \rangle = \Gamma$, we say that A *generates* Γ. If A is a finite set which generates Γ, then we say that Γ is finitely generated.

Notation. If $A = \{a_1, \ldots, a_n\}$ and A geneates Γ, then we will write $\Gamma = \langle a_1, \ldots, a_n \rangle$.

Definition. If $\exists a \in \Gamma$ such that $\Gamma = \langle a \rangle$, then we say that Γ is *cyclic*.

Remark. If Γ is cyclic, then Γ is either $\Gamma \cong \mathbb{Z}$ or $\Gamma \cong \mathbb{Z}_n$ depending on the order of the generator.

Example (Generating sets are not unique)**.** Let $\Gamma = \mathbb{Z} = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$. Then $\Gamma = \langle a^2, a^3 \rangle = \langle a^{-2}, a^5 \rangle$, etc.

Example.

(1) Finite groups are finitely generated.
(2) The free Abelian group of rank n (that is, $\mathbb{Z}^n = \langle e_1, \ldots, e_n \rangle$) is finitely generated.
(3) Finitely generated free groups are finitely generated.
(4) Gromov hyperbolic groups are finitely generated.
(5) $CAT(0)$ groups
(6) $\pi_1(M)$ for a compact manifold M
(7) $\mathrm{SL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})$.
(8) $\pi_1(K)$, where K is a finite simplicial complex
(9) Braid groups, mapping class groups
(10) $\mathrm{GL}_n(\mathbb{R})$ is NOT finitely generated. Other non-examples: $\mathbb{Q}$.

Date: Fall 2010.

Fact. If Γ is a finitely generated Abelian group, then any subgroup of Γ is also finitely generated. However, there exist finitely generated groups with subgroups which are not finitely generated.

Definition. Let S be a set, which we will call the *alphabet*. A finite sequence of elements from S is called a *word in* S. We use the notation S^* to denote the set of all words in S. A general $w \in S^*$ looks like $w = s_1 s_2 \cdots s_n$, $s_i \in S$.

Notation. Let $\overline{S}$ be a set in bijective correspondence with S. We think of the elements of this set as $\overline{S} = \{\overline{s} \mid s \in S\}$. We will use the convention that if $a \in \overline{S}$, then $a = \overline{s}$ for some $s \in S$. We let $\overline{a}$ be an alternate notation for s.

Notation. If $w \in \{S \cup \overline{S}\}^*$, then, for example, $w = s_1 s_2 \overline{s}_3 \cdots \overline{s}_k$. We use the notation $\overline{w}$ to mean:

$$\begin{aligned}\overline{w} &= \overline{\overline{s}}_k \cdots \overline{\overline{s}}_3 \overline{s}_2 \overline{s}_1 \\ &= s_k \cdots s_3 \overline{s}_2 \overline{s}_1\end{aligned}$$

Definition. Let S be a subset of the group Γ. Then $w \in \{S \cup \overline{S}\}^*$ is called *freely reduced* if it does not contain a subword of the form $x\overline{x}$ for $x \in S \cup \overline{S}$.

Example. The word $xy\overline{xy}$ is freely reduced, but $xyx\overline{x}yy\overline{y}$ is not.

We have a map, called *evaluation*, $\epsilon \colon \{S \cup \overline{S}\}^* \to \Gamma$. For example:

$$\epsilon(xy\overline{xy}) = xyx^{-1}y^{-1}.$$

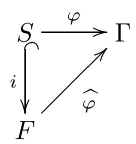
Example. If $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z} = \langle x, y \rangle$, $S = \{x, y\}$, then

$$\epsilon(xy\overline{xy}) = xyx^{-1}y^{-1} = e_\Gamma.$$

Definition. We say that Γ *is free with basis* S if $\langle S \rangle = \Gamma$, and no freely reduced word in $\{S \cup \overline{S}\}^*$ is trivial in Γ (after evaluating with ϵ).

Exercise. Free groups are torsion free.

Definition. A group F is *freely generated by a subset* $S \subset F$ if for any group Γ and any map $\varphi \colon S \to \Gamma$, there exists a unique homomorphism $\widehat{\varphi} \colon F \to \Gamma$ such that the following diagram commutes:



Theorem. *Free groups exist.*

Proof. We will construct $F(S)$, the *free group with basis* S (this is just terminology for now; it remains to show this forms a group), as the set of freely reduced words in $\{S \cup \overline{S}\}^*$ with concatenation as the operation. Our identity is the empty word, denoted e. (NB: the concatenation of two freely reduced words might not be freely reduced.)

We define an equivalence relation on $\{S \cup \overline{S}\}^*$ as follows. Two words w, w' are equivalent if they differ by a finite sequence of reductions (or expansion). Let $[w]$ be the equivalence class of w with respect to this equivalence relation. Now, take $F(S) = \{[w] \mid w \in \{S \cup \overline{S}\}^*\}$, with the operation given by $[w][w'] = [ww']$. (Check that this operation is well-defined.) The inverse is given by $[w]^{-1} = [\overline{w}]$. □

Notation. When S is finite with n elements, we write $F(S) = F_n$ for *the free group of rank n.*

Exercise. If $F_m \cong F_n$, then $m = n$.

Definition. A *graph* K is a set of vertices, denoted $V(K)$, and a set of edges, denoted $E(K)$. We will sometimes take an edge e to be an unordered pair of vertices $e = \{v, w\}$. Other times, we will want them to be ordered (*directed*), $e = (v, w)$. The distinction will be stated or clear from context. We do allow loops and multi-edges.

Definition. Recall some terminology from graph theory:

(1) *Valence* or *degree* is the number of edges coming out of a vertex.
(2) A graph is *locally finite* if all vertices have finite valence.
(3) A graph is *regular* if all vertices have the same valence.
(4) An *edge path* is an alternating sequence of vertices and edges, $\{v_0, e_1, \ldots, v_{n-1}, e_n, v_n\}$, where consecutive vertices and edges are adjacent.
(5) K is *connected* if for all $v, w \in V(K)$, there is an edge path with $v_0 = v$ and $v_n = w$.
(6) A *backtrack* is a path of the form $\{v, e, w, e, v\}$.
(7) We call a path *reduced* if it has no backtracks.
(8) A nontrivial path is a *circuit* if it starts an ends at the same vertex. A reduced circuit is called a *cycle*.

Theorem (Cayley's better theorem)**.** *Every finitely generated group can be faithfully represented as a symmetry group of a connected, directed, locally finite, regular graph.*

2. September 9, 2010

Definition. Let $\Gamma = \langle S \rangle$ be a finitely generated group with generating set S. We define a labeled, directed graph $\Delta(\Gamma, S)$ with vertices $V(\Delta) = \Gamma$. Elements g, h are connected by an edge if $g^{-1}h \in S \cup S^{-1}$. This graph is called the *Cayley graph.*

That is, for each $g \in \Gamma$ and $s \in S$, we have an edge

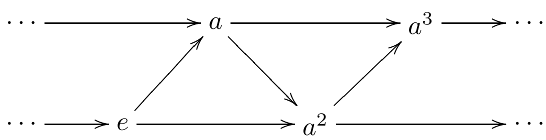
$$g \xrightarrow{s} gs$$

Right multiplication by elements of s give the edges.

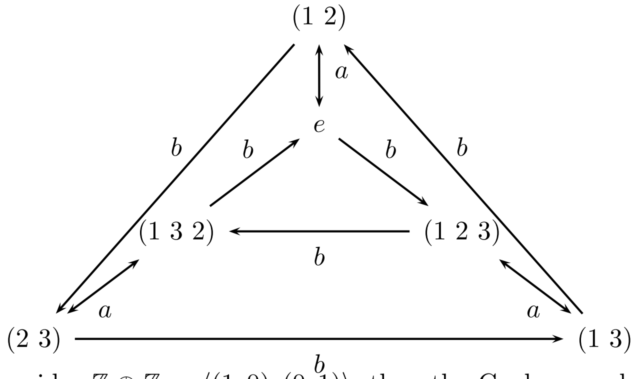
Example. Consider $\mathbb{Z} = \langle 1 \rangle$. The graph $\Delta(\mathbb{Z}, \{1\})$ looks like:

$$\cdots \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

Example. Now consider $\{a^n \mid n \in \mathbb{Z}\} = \mathbb{Z} = \langle a, a^2 \rangle$. Then the graph looks like:



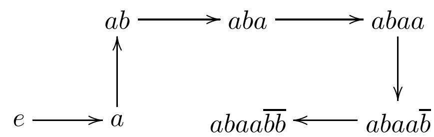
Example. Consider the symmetric group $S_3 = \langle (1\ 2), (1\ 2\ 3) \rangle = \langle a, b \rangle$. The Cayley graph is:



Example. Consider $\mathbb{Z} \oplus \mathbb{Z} = \langle (1, 0), (0, 1) \rangle$, then the Cayley graph is the integer lattice with horizontal and vertical edges.

Given $w \in \{S \cup \overline{S}\}^*$, we can draw a path in the Cayley graph that begins at e.

Example. From the previous example, take $a = (1, 0)$ and $b = (0, 1)$. Let $w = abaa\overline{b}\overline{a}$. Then the path looks like:



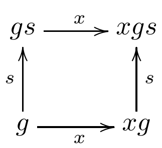
Conversely, given an edge path beginning at e in the Cayley graph, we can rad off a word in $\{S \cup \overline{S}\}^*$. This correspondence gives a bijection between words $\{S \cup \overline{S}\}^*$ and $\Delta(\Gamma, S)$. The collection of paths with no backtracking corresponds to reduced words.

Remark. Observe that if we can draw a nontrivial edge path that begins and ends at e in the Cayley graph, then there is a corresponding word in $\{S \cup \overline{S}\}^*$ that evaluates to the identity.

Corollary. *If $\Gamma = F_n$ and S is any basis, then $\Delta(F_n, S)$ is a regular $2n$-valent tree.*

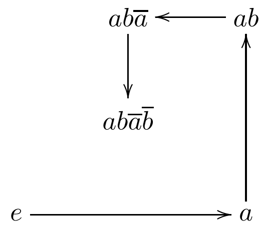
Proof. In F_n, there are no nontrivial reduced words that evaluate to e. Thus there are no cycles in $\Delta(F_n, S)$. □

Proof of Cayley's better theorem. Let $\Gamma = \langle S \rangle$ be a finitely generated group. Γ acts on the Cayley graph $\Delta(\Gamma, S)$ by left multiplication. That is, if v_g is the vertex labeled by g and $x \in \Gamma$, then $x \cdot v_g = v_{xg}$. ("x takes the vertex labeled g to the vertex labeled xg.") Consider an arbitrary edge (g, gs). We have:



Hence this action takes edges to edges, and is therefore a group of symmetries. The action is clearly faithful. Since $\Delta(\Gamma, S)$ is connected (since S generates), this completes the proof. □

Example. Consider $F_2 = \langle a, b \rangle$. (Picture of infinite 4-valent tree.) Consider the word $w = ab\overline{a}\overline{b}$. The corresponding path in the Cayley graph is:



But if the element $aba^{-1}b^{-1}$ acts on e, b^{-1} acts first, then a^{-1}, and so on. So reading off the word gives you the endpoint, but it will not correctly show the action of that element on the vertex e.

Definition. Let A be a subset of the group Γ. Then the *normal closure of* A, denoted $\langle\langle A \rangle\rangle$, is the smallest normal subgroup of Γ that contains A.

Remark. $\langle\langle A \rangle\rangle$ is characterized by the following properties:

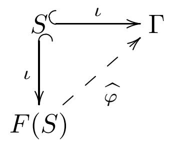
(1) $A \subset \langle\langle A \rangle\rangle$
(2) $\langle\langle A \rangle\rangle \triangleleft \Gamma$
(3) If $N \triangleleft \Gamma$ and $A \leq N$, then $\langle\langle A \rangle\rangle \subset N$.

We can also think of the normal closure as $\langle\langle A \rangle\rangle = \langle \{gag^{-1} \mid g \in G, a \in A\} \rangle$.

Remark. If A is finite, then $\langle A \rangle$ is finitely generated, but $\langle\langle A \rangle\rangle$ might not be.

Theorem. *Every group is the quotient of a free group.*

Proof. Let Γ be any group and let S be some generating set for Γ (possibly infinite). Let $F(S)$ be the free group on S. By the universal mapping property for free groups, we have



$\widehat{\varphi}$ is onto precisely because S generated Γ. By the first isomorphism theorem, we know $\Gamma \cong F(S)/\ker(\widehat{\varphi})$. □

Definition. A *presentation* of a group Γ is an isomorphism with a group of the form

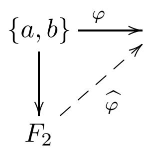
$$F/\langle\langle R \rangle\rangle$$

where F is free and $R \subset F$. If Γ is finitely generated, $F = F_n$ for some n, and R is finite, then we say that Γ is *finitely presented.*

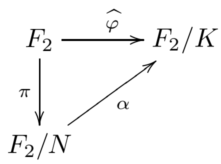
Definition. Equivalently, suppose $\Gamma = \langle S \rangle$ with S finite and $\Gamma \cong F(S)/\langle\langle R \rangle\rangle$. If R is finite, then we write $\langle S \mid R \rangle$ for the *presentation of* Γ *using* S *and* R.

Fact. $F_n = \langle a_1, \dots, a_n \mid \emptyset \rangle$ is a presentation for the free group.

Example. Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z} = \langle c, d \rangle$. Then we claim $\Gamma = \langle a, b \mid aba^{-1}b^{-1} \rangle$. We have $F_2 = \langle a, b \rangle$. By the universal mapping property:



where $\varphi(a) = c$ and $\varphi(b) = d$. Then $\Gamma \cong F_2 / \ker \widehat{\varphi}$. We want to show that $K = \langle\langle aba^{-1}b^{-1} \rangle\rangle$, which we will denote N. We have that $N \subset K$ because $cdc^{-1}d^{-1}$ is trivial in Γ. Conversely, we have:



If $x \in K$, then $\widehat{\varphi}(x)$ is trivial. Then $\alpha(\pi(x))$ is also trivial. F_2/N is Abelian (since it's a quotient by the commutator subgroup). So $\pi(x) = Na^m b^n$, hence $\alpha(\pi(x)) = Nc^m d^n = e$ implies $m = n = 0$.

3. September 14, 2010

Remark. Recall from last time: if Γ is a finitely generated group wwith generating set S, then we can construct a locally finite graph $\Delta(\Gamma, S)$ on which Γ acts by graph symmetries. If $e_\Gamma \notin S$, then Δ has no loops.

For each edge $e = g \xrightarrow{s} gs$, identify the edge isometrically with $[0, 1]$. That is, $f_e \colon [0, 1] \to \Delta$ such that $f_e(0) = g$ and $f_e(1) = gs$. We can choose these isometries equivariantly. Define a metric on Δ as follows:

$$d(p, q) = \inf\{\ell(c) \mid c \text{ is an piecewise linear path from } p \text{ to } q \text{ in } \Delta\}.$$

Definition. A *piecewise linear path* is a map $c \colon I \to \Delta$ for which there exists a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ such that

$$c_{[t_i, t_{i+1}]} = f_{e_i} \circ c_i$$

where $c_i \colon [t_i, t_{i+1}] \to [0, 1]$ is an affine map. The *length of the path* c is given by

$$\ell(c) = \sum_{i=0}^{n-1} |c_i(t_i) - c_i(t_{i+1})|$$

Remark. The function d is a metric on Δ which turns Δ into a proper, geodesic metric space. We had $\Gamma \leq \mathrm{Sym}(\Delta)$. We now have $\Gamma \leq \mathrm{Isom}(\Delta)$.

We can define a distance function on the group Γ with respect to the generating set S by taking $d_S(g, h)$ to be the length of the shortest word $w \in \{S \cup \overline{S}\}^*$ such that $\epsilon(w) = g^{-1}h$. This is called the *word metric associated to S*.

Example. Consider $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ with the standard generating set $S = \{(1, 0), (0, 1)\}$. Then $\Delta(\Gamma, S)$ is given isometrically by the integer lattice grid in $\mathbb{E}^2$ with the taxi cab metric.

Definition. Let (X, d) be a metric space. A *(unit speed) geodesic in* X is a path $\gamma\colon I = [a, b] \to M$ such that $d(\gamma(t), \gamma(u)) = |t - u|$ for all $u, t \in I$. In particular, if $\gamma(a) = x$ and $\gamma(b) = y$, then $d(x, y) = b - a$.

A *geodesic ray* is $\gamma\colon [0, \infty) \to X$ such that $d(\gamma(t), \gamma(u)) = |t - u|$ for all $u, t \in [0, \infty)$.

(X, d) is called a *geodesic metric space* if there exists a geodesic between any pair of points. It is called *uniquely geodesic* if there is only one.

Definition. A metric space (X, d) is *proper* if it's (complete and?) closed metric balls are compact.

Remark. Does the metric ball condition imply completeness?

Definition. A map $f\colon (X, d) \to (X', d')$ between metric spaces is called an *isometric embedding* if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If f is also surjective, it is called an *isometry.*

Definition. Let (X, d) and (X', d') be proper, geodesic metric spaces. A map $\varphi\colon X \to X'$ is a λ *quasi-isometric embedding* if $\lambda \geq 1$ and

$$\frac{1}{\lambda} d(x, y) - \lambda \leq d'(\varphi(x), \varphi(y)) \leq \lambda d(x, y) + \lambda.$$

If $d'(x', \varphi(X)) \leq \lambda$ for all $x' \in X'$ (equivalently, if for each $x' \in X'$ there exists $x \in X$ such that $d'(x', \varphi(x)) \leq \lambda$), then we say φ is *almost surjective.* If φ is an almost surjective quasi-isometric embedding, then φ is called a *quasi-isometry.* We write $X \sim_{QI} X'$ to mean that X is quasi-isometric with X'.

Example. (0) Any bounded set is quasi-isometric to a point.

(1) The projection map $\pi\colon R \times [0, 1] \to \mathbb{R}$ is a quasi-isometry.

(2) $\Delta(\mathbb{Z}, \{a, a^2\})$ is quasi-isometric with $\Delta(\mathbb{Z}, \{a\})$. In fact, any Cayley graph of $\mathbb{Z}$ is quasi-isometric to the real line.

(3) $\Delta(\mathbb{Z} \oplus \mathbb{Z}, \{(1, 0), (0, 1)\}) \sim_{QI} \mathbb{E}^2$. The inclusion map $\iota\colon \Delta \to \mathbb{E}^2$ is almost surjective. We have

$$\frac{1}{\sqrt{2}} d_\Delta(x, y) \leq d_{\mathbb{E}^2}(x, y) \leq d_\Delta(x, y)$$

So we can take $\lambda = 2$, for example, and we have a quasi-isometry.

(4) If $\Gamma = \langle S \rangle = \langle S' \rangle$ for finite generating sets S, S', then $\Delta(\Gamma, S) \sim_{QI} \Delta(\Gamma, S') := \Delta'$. We take $f\colon \Delta \to \Delta'$ where $f_{V(\Delta)}$ is the identity. Take $\lambda_1 = \max\{d'(e, a) \mid a \in S\}$ and $\lambda_2 = \max\{d(e, a') \mid a' \in S'\}$. Take $\lambda = \max\{\lambda_1, \lambda_2\}$. Then

$$\frac{1}{\lambda_2} d(g, h) \leq d'(g, h) \leq \lambda_1 d(g, h)$$

4. September 16, 2010

Remark. The definition of quasi-isometry makes sense for any metric space. We did not need to assume proper or geodesic.

Let $\Gamma = \langle S \rangle$ be a finitely generated group, let $\Delta = \Delta(\Gamma, S)$ be the Cayley graph, and let Γ_S be the group with the word metric. The inclusion map $i\colon \Gamma_S \to \Delta$ is an isometric embedding that is almost onto with $\lambda = 1$, hence i is a quasi-isometry. If S, S' both generate Γ, then $\Gamma_S \sim_{QI} \Gamma_{S'}$. We have $\Delta \sim_{QI} \Gamma_S$ and

$\Gamma_{S'} \sim_{QI} \Delta' = \Delta(\Gamma, S')$, so $\Delta \sim_{QI} \Delta'$. (We will prove as an exercise that $\sim_{QI}$ is transitive, and we will show in the following proposition that it is reflexive.)

Proposition. *Suppose $\varphi\colon X \to Y$ is a quasi-isometry with constant λ. Then there exists $\psi\colon Y \to X$ which is a quasi-isometry (with constant Λ) such that*

$$d(\psi \circ \varphi, 1_X) \leq \Lambda$$
$$d(\varphi \circ \psi, 1_Y) \leq \Lambda$$

(Without loss of generality, we will take $\Lambda \geq \lambda$ at least.)

Proof. Suppose $\varphi\colon X \to Y$ is a λ-QI. Let $y \in Y$. There exists $x \in X$ such that $d(y, \varphi(x)) \leq \lambda$. Define $\psi(y) = x$. Now $d(\varphi \circ \psi(y), y) \leq \lambda$ by definition of ψ. Also, let $x' = \psi \circ \varphi(x)$. Then

$$\frac{1}{\lambda} d(x, x') - \lambda \leq d(\varphi(x), \varphi(x')) \leq \lambda$$

hence $d(x, x') \leq 2\lambda^2$.

Now let $x \in X$. We will find $y \in Y$ such that $d(x, \psi(y))$ is bounded independent of x as follows. Pick $y = \varphi(x)$.

It remains to show the quasi-isometry inequality:

$$\frac{1}{\Lambda} d(y, y') - \Lambda \leq d(\psi(y), \psi(y')) \leq \Lambda d(y, y') + \Lambda.$$

Let $x = \psi(y)$ and $x' = \psi(y')$. From the QI inequality for φ, we have

$$\begin{aligned}\frac{1}{\lambda} d(x, x') - \lambda &\leq d(\varphi(x), \varphi(x')) \\ &\leq \lambda[d(\varphi(x), \varphi(x')) + \lambda] \\ &\leq \lambda[d(y, y') + 2\lambda] + \lambda^2 \\ &\leq \lambda d(y, y') + 3\lambda^2.\end{aligned}$$

Now we need

$$\frac{1}{\Lambda} d(y, y') - \Lambda \leq d(x, x').$$

Use $d(\varphi(x), \varphi(x')) \leq \lambda d(x, x') + \lambda$. By the reverse triangle inequality:

$$d(\varphi(x), \varphi(x')) \geq d(y, y') - 2\lambda.$$

So we can take $\Lambda \geq 3\lambda^2$, and the result follows. □

Fact. QI is an equivalence relation on metric spaces, and we use the notation $X \sim_{QI} Y$.

Exercise. Only Charlie: The empty set is only QI to itself.

Fact.
(1) $\mathbb{E}^1 \nsim_{QI} [0, 1]$. In fact, being bounded is a QI invariant.
(2) $\mathbb{E}^1 \nsim_{QI} [0, \infty)$.
(3) $\mathbb{E}^2 \nsim_{QI} \mathbb{E}^1$.
(4) Let T_n be the regular tree of valence n. $T_3 \nsim_{QI} \mathbb{E}^1$ and $T_3 \nsim_{QI} \mathbb{E}^2$.
(5) $T_n \sim_{QI} T_m$ for all $n, m \geq 3$. (This is nontrivial.)
(6) $\mathbb{E}^n \sim_{QI} \mathbb{E}^m$ implies $n = m$.

Remark. For T_3, T_4, if you collapse edges carefully in T_3, you will get T_4. Then show that this map is a quasi-isometry.

Big theorem coming up:

Theorem. *If X, Y are proper, geodesic metric spaces, then any QI $f: X \to Y$ will induce a homeomorphism $f_\epsilon: \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$.*

Let Γ, Γ' be finitely generated groups. When can we say that $\Gamma \sim_{QI} \Gamma'$? (We can write $\Gamma \sim_{QI} \Gamma'$ without reference to a generating set, since QI is independent of the choice of generating set.) Guess: maybe they contain isomorphic finite index subgroups.

Definition. Two groups Γ, Γ' are *commensurable* if they contain isomorphic finite index subgroups.

Fact. Being commensurable is an equivalence relation. A big part of geometric group theory looks at the relationship between commensurability and quasi-isometry.

If $\Gamma \sim_C \Gamma'$, then $\Gamma \sim_{QI} \Gamma'$. The converse is not always true. A major question is: under what conditions does QI imply commensurability. This is called the QI rigidity question.

Theorem. *There exist M, N closed, hyperbolic 3-manifolds with:*

(1) *no common finite cover*
(2) $\pi_1(M) \sim_{QI} \pi_1(N)$ *(and both are QI to $\mathbb{H}^3$).*

Example. Consider $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$, $\varphi \in \mathrm{GL}(2, \mathbb{Z})$. The QI class is determined by the matrix for φ. (Bridson)

Theorem (Svarc-Milner)**.** *Suppose Γ acts cocompactly and properly discontinuously by isometries on a proper, geodesic metric space X. Then Γ is finitely generated, and the map $\Gamma \to X$ given by $\gamma \mapsto \gamma \cdot x_0$ is a QI for any choice of base point $x_0 \in X$.*

Definition. Let (X, d) be a metric space. $\mathrm{Isom}(X)$ is a group. Let Γ be a group and $\Phi: \Gamma \to \mathrm{Isom}(X)$ a homomorphism. Then we say Γ *acts on X by isometries.* We write $\Phi(\gamma)(x) =: \gamma \cdot x$. The action is *faithful* if Φ is injective. The action is *free* if for all $x \in X$ and for all $\gamma \neq e$, then $\gamma \cdot x \neq x$. The action is called *cocompact* if there exists a compact set $K \subset X$ such that $\Gamma \cdot K = X$. The action is called *properly discontinuous* if for all compact $K \subset X$, the set $\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\}$ is finite.

If $G \leq \Gamma$ is a finite index subgroup of the finitely generated group Γ, and if $X = \Delta(\Gamma, S)$ for any finite generating set S, then $G \curvearrowright X$ properly discontinuously and cocompactly by isometries.

5. September 21, 2010

Theorem (Śvarc–Milnor)**.** *Suppose Γ is a group acting "geometrically" (i.e. properly discontinuously and cocompactly by isometries) on a proper geodesic metric space X. Then Γ is finitely generated, and for any choice of basepoint $x_0 \in X$, the map $\Gamma \to X$ given by $\gamma \mapsto \gamma \cdot x_0$ is a QI.*

Theorem. *Let X be a topolocial space, and let Γ act on X by homeomorphisms. Suppose there exists an open set $U \subset X$ such that $\Gamma \cdot U = X$. Then if X is connected, then $S = \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$ generates Γ.*

Moreover, if U, X are both path-connected and X is simply connected, then

$$R = \{s_1 s_2 s_3^{-1} \mid s_i \in S, U \cap s_1 U \cap s_3 U \neq \emptyset, s_1 s_2 = s_3\}$$

forms a set of relators for Γ.

Proof. We will only prove the first claim. Let $H = \langle S \rangle \leq \Gamma$. Let $V = H \cdot U$ and $V' = (\Gamma \setminus H) \cdot U$. If $V \cap V' \neq \emptyset$, then we have $hu = h'u'$. It follows that $u = h^{-1}h'u'$, hence $h^{-1}h'U \cap U \neq \emptyset$ and $h^{-1}h' \in S$. This means $h' \in HS \subset H$. But $h' \notin H$ by assumption, a contradiction. So $V \cap V' = \emptyset$. Since X is connected, $V' = \emptyset$, hence $\Gamma \setminus H = \emptyset$ and $\Gamma = H$. ☐

Theorem. *Suppose Γ is a group. Γ is finitely presented if and only if Γ acts properly discontinuously and cocompactly by homeomorphisms on a simply connected geodesic space.*

This was all a topological aside. For the Śvarc–Milnor theorem, we'll need to think more metrically.

Proof of Śvarc–Milnor. Let $C \subset X$ be a compact set such that $\Gamma \cdot C = X$. Choose $x_0 \in X$ and $D > 0$ such that $C \subset B(x_0, D/3)$ and let

$$A = \{\gamma \in \Gamma \mid \gamma \cdot B(x_0, D) \cap B(x_0, D) \neq \emptyset\}.$$

Since the action is properly discontinuous, A is finite.

We have a map $f\colon \Gamma_A \to X$ given by $\gamma \mapsto \gamma \cdot x_0$. We need to show f is a QI. We know that every point of X is within $D/3$ of an orbit point. So f is almost surjective. (The more common phrase is: *the image is quasi-dense.*) We must show that

$$\frac{1}{\lambda} d_A(e, \gamma) - \lambda \leq d_X(f(e), f(\gamma)) \leq \lambda d_A(e, \gamma) + \lambda.$$

(Since both metrics are equivariant, we may assume we're at the origin.) Take $\lambda = \max\{d_X(x_0, a \cdot x_0) \mid a \in A\}$. Suppose $d_A(e, \gamma) = n$. Then $\gamma = a_1 a_2 \cdots a_n$, $a_i \in A$. Let $g_i = a_1 \cdots a_i$. Then

$$d_X(g_i x_0, g_{i+1} x_0) = d_X(x_0, g_i^{-1} g_{i+1} x_0) \leq \lambda.$$

Hence $d_X(x_0, \gamma x_0) \leq \lambda n = \lambda d_A(e, \gamma)$.

For the other inequality, we can consider a geodesic from x_0 to γx_0 as an isometry $c\colon [0, d(x_0, \gamma x_0)] \to X$. Consider the points $c(iD/3)$, and for each one choose an orbit point $\gamma_i x_0$ within distance $D/3$. Then $d(c(iD/3), c((i+1)D/3)) \leq D/3$, $d(\gamma_i x_0, \gamma_{i+1} x_0) \leq D$, and $d(x_0, \gamma x_0) \geq (n-1)D/3 \geq d(e, \gamma) - 1$. We have $\gamma = \gamma_0(\gamma_0^{-1}\gamma_1) \cdots (\gamma_{n-1}^{-1}\gamma_n)$, where $\gamma_0 = e$ and $\gamma_i^{-1}\gamma_{i+1} \in A$. So for this inequality, choose $\lambda = 1$, and for the overall proof, take the maximum λ. ☐

As a useful application of this theorem, if we want to show that two groups are QI, we can make them both act geometrically on the same space.

Example. Let M, N be compact hyperbolic manifolds. Their fundamental groups are QI, because they both act on $\mathbb{H}^n$ geometrically.

Corollary. *If Γ is a finitely generated group and G a subgroup of finite index, then G is finitely generated and $G \sim_{QI} \Gamma$.*

Proof. G acts geometrically on any Cayley graph for Γ. (The generating set doesn't matter, since all the Cayley graphs are QI.) ☐

Definition. For any property P, we say Γ is *virtually* P if there is some finite index subgroup $G \leq \Gamma$ which has property P.

Question: when does $\Gamma \sim_{QI} \Gamma'$ imply $\Gamma \sim_C \Gamma'$? (We already know the converse is always true by the previous theorem.)

Fact. In the following cases, $\Gamma \sim_{QI} \Gamma'$ implies $\Gamma \sim_C \Gamma'$:

(0) Γ or Γ' is finite.

(1) Γ or Γ' is virtually $\mathbb{Z}$.

(2) If both Γ and Γ' are finitely generated and virtually Abelian, then they are both virtually $\mathbb{Z}^n$ for some n: let Γ be virtually Abelian. Then there exists a finite index group $G \leq \Gamma$, which is finitely generated. By the structure theorem, $G \cong H \times T$, where T is the torsion part and $H \cong \mathbb{Z}^n$ for some n. For QI purposes, we do not lose generality by assuming G is torsion free. Similarly, we'll have a $G' \leq \Gamma'$ which is $\mathbb{Z}^m$ for some m. So we really need to show that $\mathbb{Z}^n \sim_{QI} \mathbb{Z}^m$ implies $n = m$.

It's easier to prove that $\mathbb{E}^n \sim_{QI} \mathbb{E}^m$ implies $n = m$. (The claim follows from Švarc–Milnor.) To show this, we would start by showing $\mathbb{R}^2 \sim_{QI} \mathbb{R}$.

(3) If Γ is a finitely generated group and $\Gamma \sim_{QI} \mathbb{Z}^n$, then Γ is virtually $\mathbb{Z}^n$.

Proposition. *If Γ is a finitely generated group and $\Gamma \sim_{QI} \mathbb{Z}$, then Γ is virtually $\mathbb{Z}$.*

6. September 23, 2010

Recall from last time:

Corollary (Corollary to Gromov's theorem)**.** *If Γ is a finitely generated group and $\Gamma \sim_{QI} \mathbb{Z}^n$ then Γ is virtually $\mathbb{Z}^n$.*

Corollary. *If Γ is a group of polynomial growth, then Γ is virtually nilpotent.*

Continuing our examples from last time:

Fact. (4) Any finitely generated group Γ that is QI to $\mathbb{H}^n$ can be realized as a finite extension of a proper cocompact subgroup of $\text{Isom}(\mathbb{H}^n)$. That is:

$$1 \to F \to \Gamma \to \Gamma' \to 1$$

where Γ' is a proper cocompact subgroup of $\text{Isom}(\mathbb{H}^n)$ and F is finite. (Result of Tukia/Gabai, Casson–Jungreis.)

Remark. (E. Rieffel) If $\Gamma \sim_{QI} \mathbb{H}^2 \times \mathbb{R}$ for a finitely generated group Γ, then

$$1 \to F \to \Gamma \to \Gamma' \to 1$$

where Γ' is a proper cocompact subgroup of either $\text{Isom}(\mathbb{H}^2 \times \mathbb{R})$ or $\text{Isom}(\text{PSL}(2, \mathbb{R}))$.

Proposition (Special case of Borsak–Ulam theorem)**.** *Suppose $g \colon S^1 \to \mathbb{E}^1$ is a continuous map. Then there exists an antipodal pair $x = (\cos t_0, \sin t_0)$ and $-x = (-\cos t_0, -\sin t_0)$ so that $g(x) = g(-x)$.*

Proof. Let $f \colon [0, \pi] \to \mathbb{E}^1$ be given by

$$f(t) = g(\cos t, \sin t) - g(-\cos t, -\sin(t)).$$

Now we have $f(0) = g(1, 0) - g(-1, 0)$ and $f(\pi) = g(-1, 0) - g(1, 0)$. If $f(0) = 0 = f(\pi)$, then the antipodal pair is $(1, 0), (-1, 0)$. Otherwise, $f(0)$ and $f(\pi)$ have opposite sign. By the intermediate value theorem, we're done. □

Proposition. $\mathbb{E}^2 \not\sim_{QI} \mathbb{E}^1$.

Proof. Suppose $\varphi\colon \mathbb{E}^2 \to \mathbb{E}^1$ is a QI with constant λ. Choose n large enough, and take $2n$ equally spaced points $x_0, x_1, \ldots, x_{2n} = x_0$ on the circle of radius n, with x_i and x_{i+n} antipodal, and having $d_{\mathbb{E}^2}(x_i, x_{i+1}) \leq \pi$. Since φ is a QI, $|\varphi(x_i) - \varphi(x_{i+1})| \leq \lambda\pi + \lambda$. Define a map $f\colon S^1 \to \mathbb{E}^1$ by $f(x_i) = \varphi(x_i)$, and the arc between x_i and x_{i+1} onto the interval between $\varphi(x_i)$ and $\varphi(x_{i+1})$. there exists an antipodal pair $x, -x$ so that $f(x) = f(-x)$.

Choose an x_i that is closest to x. Then $d(x, x_i) \leq \pi$ and $d(-x, -x_i) \leq \pi$. Now $|f(x) - f(x_i)| \leq \lambda\pi + \lambda$, and $|f(-x) - f(-x_i)| \leq \lambda\pi + \lambda$. Now we have

$$\begin{aligned}|\varphi(x_i) - \varphi(-x_i)| &= |f(x_i) - f(-x_i)| \\ &\leq |f(x_i) - f(x)| + |f(x) - f(-x_i)| \\ &\leq 2(\lambda\pi + \lambda).\end{aligned}$$

Hence $d(x_i, -x_i)$ is bounded independent of n by the other QI inequality. But we know this distance to be $2n$, so we can choose n large enough that this is a contradiction. (Exercise: figure out how large n needs to be.) $\square$

Exercise. For homework, $\mathbb{E}^1 \not\sim_{QI} [0, \infty)$.

Lemma (Ping-pong lemma)**.** *Suppose Γ is a group generated by a set $S = \{s_1, \ldots, s_n\}$. Suppose Γ acts on a set X. That is, we have a homomorphism $\varphi\colon \Gamma \to \mathrm{Sym}(X)$. Suppose we can choose $2n$ subsets of X, $X_1^+, X_2^+, \ldots, X_n^+, X_1^-, \ldots, X_n^-$, one for each of s_i and s_i^{-1}, which are nonempty, pairwise disjoint subsets of X such that $s_i(X - X_i^-) \subsetneq X_i^+$ and $s_i^{-1}(X - X_i^+) \subsetneq X_i^-$. Then $\Gamma \cong F(S)$.*

Proof. For simplicity, suppose there exists $p \in X \setminus \bigcup_{i=1}^n (X_i^+ \cup X_i^-)$. We need to show that if w is a nontrivial reduced word in $\{S \cup \overline{S}\}^*$, then $\varepsilon(w) \neq e_\Gamma$. We will show that $w(p) \neq p$ (abusing notation), so that $\varphi(w) \neq 1_X$.

Claim. *Suppose $w = a_1^{\epsilon_1} \cdots a_k^{\epsilon_k}$, $a_i \in S$, $\epsilon = \pm 1$. Then $w(p) \in X_j^+$ if $\epsilon_1 = 1$ or $w(p) \in X_j^-$ if $\epsilon_1 = -1$, where $a_1 = s_j$.*

Proof. We will induct on the length k of the reduced word w. Suppose $k = 1$. Then $w = s_j$ or $w = s_j^{-1}$ for some j. Without loss of generality, suppose $w = s_j$. Since $p \notin X_j^-$, $w(p) \in X_j^+$, proving the base case.

Now suppose the claim is true for reduced words of length up to $k - 1$. Suppose without loss of generality that $a_1^{\epsilon_1} = s_j$. We want to show $w(p) \in X_j^+$. Suppose not. Then $s_j^{-1}(w(p)) \in X_j^{-1}$. But $s_j^{-1} w = a_2^{\epsilon_2} \cdots a_k^{\epsilon_k}$ has length $k - 1$, and $a_2^{\epsilon_2} \neq s_j^{-1}$ since w was reduced. By induction, $s_j^{-1} w(p) \notin X_j^-$ (because the $X_i^\pm$ are pairwise disjoint), thus the claim is proven. $\square$

By the claim, then $w(p) \in X_i^\pm$, but p is not in any of them. Hence $\varphi(w) \neq 1_X$, and every freely reduced word is nontrivial. $\square$

Here's one application. Let Γ be an interesting group (like the mapping class group of some surface). Take a finite set of elements $f_1, \ldots, f_k$ in Γ, and ask: does there exists an N_0 so that for all $N \geq N_0$, $\langle f_1^N, \ldots, f_k^N \rangle \cong F_k$?

Open Question. *Is the Burau representation faithful? Equivalently, do a particular pair of matrices in $\mathrm{GL}_3(\mathbb{Z}[t, t^{-1}])$ generate an F_2?*

Example. Let $\Gamma = F_2$ and $X = \Delta(\Gamma, \{a, b\})$. The origin with the four half-edges leaving it forms a fundamental domain. Removing this domain, we are left with four maximal subtrees. These will be the $X^{\pm}_{a,b}$.

Let $H \leq F_2 = \langle a, b \rangle$ be the subgroup consisting of even length words. We want to show that $H \cong F_3$. We have a fundamental domain $\mathscr{F}$ (Charlie has the picture). After removing it, there are six connected components remaining. We have

$$\{g \in H \mid g \cdot \mathscr{F} \cap \mathscr{F} \neq \emptyset\} = \{a^2, a^{-2}, ab, b^{-1}a^{-1}, ab^{-1}, ba^{-1}\}.$$

So $S = \{a^1, ab, ab^{-1}\}$ is a free basis for H.

7. September 28, 2010

Finishing a thought from last time:

Theorem. *A group Γ is free if and only if Γ acts freely on a tree.*

Corollary. *Every subgroup of a free group is free.*

Example. Let $\Gamma = \mathbb{Z}_3 \star \mathbb{Z}_4$. Then Γ acts on a tree but not freely. However, Γ is virtually free. Let $T_{3,4}$ be the infinite tree constructed as follows. Start with a valence 4 vertex. Each of its neighbors has valence 3. Each of their neighbors has valence 4, and so on. Suppose $\mathbb{Z}_3 = \langle a \rangle$ and $\mathbb{Z}_4 = \langle b \rangle$. Then the action of b on $T_{3,4}$ is the rotation about the center vertex, and the action of a is rotation about, say, the vertex north of the center.

Remark. The free product $\mathbb{Z}_3 \star \mathbb{Z}_4$ is given by

$$\{a^{i_1} b^{j_1} \cdots a^{i_n} b^{j_n} \mid 1 \leq i_k \leq 2, 1 \leq j_k \leq 3, 0 \leq i_1 \leq 2, 0 \leq j_n \leq 3\}.$$

Consider $\varphi\colon \mathbb{Z}_3 \star \mathbb{Z}_4 \to \mathbb{Z}_3 \oplus \mathbb{Z}_4$. Then $\ker \varphi$ is free.

Fact.
1. Any finite group acting on a tree will fix a point.
2. The free product of two finite groups is virtually free.

Recall: if Γ is a group and $S \subset \Gamma$ is any subset, we can build a Cayley graph $\Delta = \Delta(\Gamma, S)$. We have seen that Δ is connected if and only if S generates.

Theorem. *Let $\Gamma = \langle S \rangle$, S finite. Thus $F(S) \twoheadrightarrow \Gamma$. Let N be the kernel of this map, and let $R \subset N$. Consider the 2-complex obtained by attaching 2-cells to all edge loops in $\Delta(\Gamma, S)$ that are labeled by $r \in R$. The resulting 2-complex is simply connected if and only if $\langle\langle R \rangle\rangle = N$.*

Proof. Consider $T = \Delta(F(S), S)$. This is a tree. Then we can view $\Delta(\Gamma, S)$ as the quotient of T by the action of N. A word $w \in \{S \cup \overline{S}\}^*$ defines an element of N if and only if it is the label of an edge loop in $\Delta(\Gamma, S)$ at e_Γ. (That is, we have an identification $\pi_1(\Delta, e_\Gamma) \cong N$.)

Let $u \in F(S)$ be any reduced word, $r \in R$ fixed, and $\varepsilon(u)$ be the evaluations of the words in Γ. Then r traces out an edge loop at each $\varepsilon(u)$ in $\Delta(\Gamma, S)$. We attach a 2-cell along the word r at every $\varepsilon(u)$. Then the resulting complex has fundamental group given by $N/\langle\langle u^{-1} r u \rangle\rangle$.

If we attach a 2-cell along every such r-labeled edge loop for $r \in R$, then the resulting 2-complex K has $\pi_1(K) \cong N/\langle\langle R \rangle\rangle$. The result follows. □

Proposition. *Suppose (Γ_1, A_1) and (Γ_2, A_2) are finitely generated groups and $\Gamma_2 = \langle A_2 \mid R_2 \rangle$ is a finite presentation. If $\Gamma_1 \sim_{QI} \Gamma_2$ with constant λ, then Γ_1 is also finitely presented with generating set A_1.*

Proof. Let ρ be the longest $r \in R_2$. Let K_2 be the 2-complex obtained by attaching 2-cells along any loop of length at most ρ in $\Delta(\Gamma_2, A_2)$. By the previous theorem this complex will be simply connected.

Claim. *There exists an $M = M(\rho, \lambda)$ such that if we attach 2-cells along all loops of length at most M in $\Delta(\Gamma_1, A_1)$, then the resulting 2-complex is simply connected.*

Proof. Let the resulting complex be K_1. Consider a loop $\varphi\colon \partial D \to K_1$ with vertices $g_1, g_2, \ldots, g_n$. Let v_i be the preimage of g_i in D. If f is the QI from Γ_1 to Γ_2, then $f(g_i)$ give some collection of vertices in K_2. We will define a new map $\alpha\colon \partial D \to K_2$ so that $\alpha(v_i) = f(g_i)$. Then we extend to a map $\widehat{\alpha}\colon D \to K_2$. We sketch the remainder of the proof.

Divide the resulting disk in K_2 into smaller disks of boundary length at most ρ. Use the QI inequalities to pull these back to K_1 to show that $\varphi(\partial D)$ bounds a disk in K_1. M will be determined by ρ and the QI inequalities. The claim will follow by construction of K_1. □

The claim finishes the proposition. □

We are working up to the following theorem.

Theorem (mostly Hopf, 1940s)**.** *Let Γ be a finitely generated group. Then:*

(1) *Γ has $0, 1, 2$ or ∞ many ends.*
(2) *Γ has 0 ends if and only if Γ is finite.*
(3) *Γ has 2 ends if and only if Γ is virtually $\mathbb{Z}$.*
(4) *$\mathrm{End}(\Gamma)$ is compact. If it's infinite, theen it is uncountable and each point is an accumulation point.*
(5) *(Stallings) Γ has infinitely many ends if and only if Γ splits as $\Gamma \cong A \star_C B$ or $\Gamma \cong A \star_C$ for a finite group C, and $|A/C| \geq 3, |B/C| \geq 2$.*

Remark. $A \star_C B$ is the *amalgamated free product of A and B*. $A\star_C$ is the *HNN extension.*

Definition. Let Γ be a group, and let $\varphi\colon H \to K$ be an isomorphism between subgroups of Γ. The *HNN extension* of Γ relative to φ is

$$\Gamma \star_\varphi = \langle S, t \mid R, tht^{-1} = \varphi(h), h \in H\rangle.$$

The new generator t is called the *stable letter.*

Example. (1) $\mathbb{Z}^n$ is 1-ended for $n \geq 2$.
(2) F_n is infinite-ended for $n \geq 2$.
(3) F_1 is 2-ended.

Definition. Recall that a continuous map $f\colon X \to Y$ between topolocial spaces is called *proper* if $f^{-1}(C)$ is compact for all compact $C \subset Y$.

Proposition. *Let $x_0 \in X$ be a fixed base point in a metric space X. Consider the function $X \to \mathbb{R}$ given by $x \mapsto d(x, x_0)$. Then X is proper if and only if this function is proper.*

Definition. Let X be a topological space. A *ray* in X is a continuous map $r\colon [0, \infty) \to X$, and a *proper ray* is a ray which is proper.

Definition. Suppose $r_1, r_2\colon [0, \infty) \to X$ are proper rays. Then we say r_1 *and* r_2 *converge to the same end* if for all compact $C \subset X$ there exists $N > 0$ such that $r_1([N, \infty))$ and $r_2([N, \infty))$ lie in the same path component of $X \setminus C$.

Notation. Converging to the same end gives an equivalence relation on the set of proper rays. We write $end(r) = [r]$ for the equivalence class of a ray r.

Definition. The *ends of* X are the classes of proper rays. We write $\text{Ends}(X) = \{end(r)\}$, where r ranges over all proper rays.

Definition. We say r_n *converges to* r, written $end(r_n) \to end(r)$, if for all compact $C \subset X$, there exist $\{N_n\}$ such that $r_n([N_n, \infty))$ and $r([N_n, \infty))$ are in the same path component of $X \setminus C$ for n sufficiently large.

We give a topology to $\text{Ends}(X)$ as follows. A subset $B \subset \text{Ends}(X)$ is closed if $end(r) \in B$ whenever $\{end(r_n)\} \subset B$ converges to r.

8. September 30, 2010

If X is a proper geodesic metric space, then any compact set C is contained in some open ball $B(x_0, R)$, which is contained in the closed ball $\overline{B(x_0, R)}$. So we may replace compact sets with open balls in the definition of ends.

Definition. Let $k > 0$. A *k-path in X connecting x to y* is $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq k$.

Lemma. *Let $k > 0$, and let r_1, r_2 be proper rays.*

1. *$end(r_1) = end(r_2)$ if and only if for all $R > 0$ there exists a $T > 0$ such that $r_1(t)$ can be joined to $r_2(t)$ by a k-path in $X \setminus B(x_0, R)$ for all $t > T$.*
2. *Let $\mathscr{G}_{x_0}$ be the set of all geodesic rays in X emanating from x_0. There is a map $\mathscr{G}_{x_0} \to \text{Ends}(X)$ given by $c \mapsto end(c)$. This map is surjective.*

Proof.

1. Suppose we have a k-path from $r_1(t)$ to $r_2(t)$ in $X \setminus B(x_0, R)$. Join successive points by geodesics to get a path in $X \setminus B(x_0, R - k)$. So $r_1(t)$ and $r_2(t)$ are in the same path component of the complement of a ball. In a proper geodesic metric space, since we can replace compact sets by open balls in the definition of ends, $end(r_1) = end(r_2)$. The converse is trivial.
2. Let $end(r) \in \text{Ends}(X)$, with $r\colon [0, \infty) \to X$ a proper ray. Without loss of generality (by connecting $r(0)$ with a geodesic to x_0) we may assume $r(0) = x_0$.

   Let $c_n\colon [0, d_n] \to X$ bee a geodesic joining x_0 to $r(n)$. Extend each c_n by the constant map so that $c_n\colon [0, \infty) \to X$ and $c_n(t) = d_n$ for $t \geq d_n$. The collection $\{c_n\}$ is an equicontinuous family of functions. Apply Arzelá–Ascoli to obtain a subsequence that converges to a geodesic (by construction) map $c\colon [0, \infty) \to X$.

□

Theorem. *Let X_1, X_2 be proper geodesic metric spaces and $f\colon X_1 \to X_2$ a QI with constant λ. Then f induces a homeomorphism $f_\varepsilon\colon \text{Ends}(X_1) \to \text{Ends}(X_2)$.*

Proof. Let $end(r) \in \text{Ends}(X_1)$. Without loss of generality, assume r is a geodesic. Connect the images $f(r(i))$ by geodesics. This creates a proper ray in X_2 (which follows from the fact that f is a QI, but there is a little bit to show here). Denote this ray by $f_*(r)$. Then define $f_\varepsilon(end(r)) = end(f_*(r))$.

This map is well-defined because k-paths in X_1 turn become $(\lambda k + \lambda)$-paths in X_2. By the lemma, this map is defined on the whole domain. f_ε is continuous, but we won't do this because there's a little more work to do. Use the QI to find an inverse map which shows f_ε is bijective, and therefore a homeomorphism. □

Definition. Let Γ be a finitely generated group and $\Delta = \Delta(\Gamma, S)$ any Cayley graph. Then we define $\text{Ends}(\Gamma) = \text{Ends}(\Delta)$.

Theorem. *Let Γ be a finitely generated group. Then Γ has 0, 1, 2 or infinitely many ends.*

Proof. Γ acts by homeomorphisms on $\text{Ends}(\Gamma)$. (Recall that Γ acts by isometries on $\Delta(\Gamma, S)$.) Thus we have a homomorphism $\varphi\colon \Gamma \to \text{Homeo}(\text{Ends}(\Gamma))$.

If $\text{Ends}(\Gamma)$ is finite, then $\text{Homeo}(\text{Ends}(\Gamma))$ is finite, so $K = \ker\varphi$ has finite index in Γ. By quasi-density, there exists μ such that for any $\gamma \in \Gamma$, there exists $k \in K$ with $d(\gamma, k) \leq \mu$.

Suppose there are three distinct ends e_0, e_1, e_2. Draw geodesic rays r_1, r_2 at e_Γ with $end(r_i) = e_i$ for $i = 1, 2$. Take a geodesic ray r_0' with $end(r_0') = e_0$. Define a proper ray r_0 with $end(r_0) = e_0$ such that $d(r_0(n), e_\Gamma) \geq n$, $r_0(n) \in K$, and $r_0(n)$ is at least μ-close to r_0'. r_0 is a proper ray.

Choose ρ so that $r_0([\rho, \infty))$, $r_1([\rho, \infty))$, and $r_2([\rho, \infty))$ are in different path components of $X \setminus B(e_\Gamma, \rho)$. If $t, t' \geq 2\rho$, then $d(r_1(t), r_2(t')) \geq 2\rho$.

Choose $n > 3\rho$. Then $\gamma_n = r_0(n)$ is in K, and $d(e_\Gamma, \gamma_n) > e\rho$ by construction of r_0. Since its in K, γ_n acts like the identity on $\text{Ends}(\Gamma)$, and γ_n acts like an isometry on Δ. So $\gamma_n(r_1)$ is a geodesic ray based at γ_n. But $end(\gamma_n(r_1)) = e_1$ since γ_n acts trivially on $\text{Ends}(\Gamma)$. Since γ_n lies in a different path component of $X \setminus B(e_\Gamma, \rho)$, there must be some t such that $\gamma_n(r_1(t)) \in B(e_\Gamma, \rho)$. Since $n > 3\rho$, $t > 2\rho$.

Similarly, there exists $t' > 2\rho$ so that $\gamma_n(r_2(t')) \in B(e_\Gamma, \rho)$. so that

$$d(\gamma_n(r_1(t)), \gamma_n(r_2(t'))) \leq 2\rho.$$

Since γ_n acts like an isometry, $d(r_1(t), r_2(t')) \leq 2\rho$, a contradiction. So Γ cannot have three distinct ends. □

9. October 5, 2010

Consider $\text{Ends}(X)$. Recall that we defined $end(r_n) \to end(r)$ to mean that for all compact $C \subset X$, there exist $\{N_n\}$ such that $r_n([N_n, \infty))$ and $r([N_n, \infty))$ lie in the same path component of $X \setminus C$ for sufficiently large n.

Proof continued from last time. If $f\colon X_1 \to X_2$ is a QI, let $f_\varepsilon\colon \text{Ends}(X_1) \to \text{Ends}(X_2)$ be the induced map. To show continuity, we must show that if $end(r_n) \to end(r)$ in X_1, then $f_\varepsilon(end(r_n)) \to f_\varepsilon(end(r))$.

Claim. *If r_n and r are geodesic rays with $r_n([N, \infty))$ and $r([N, \infty))$ in the same path component of $X_1 \setminus B(x_0, M)$, then there exists $\widetilde{M}$ so that $f_*(r_n)$, $f_*(r)$ from N to ∞ are in the same path component of $X_2 \setminus B(x_0, \widetilde{M})$.*

Proof. Let p be a point on r_n outside a ball of radius M, and let q be a point in r outside a ball of radius M. Suppose $f(p)$ is on $f_*(r_n)$ and $f(q)$ is on $f_*(r)$. Then $d(f(x_0), f(p) \geq \frac{1}{\lambda}M - \lambda$. Partition a geodesic path from $f(p)$ to $f(q)$ with points γ_i which are each distance at most 1. Then $f(\gamma_i)$ are at most 2λ apart. Connecting these by geodesics and using the lemma from last time, if we take $\widetilde{M} = \frac{1}{\lambda}M - 3\lambda$, the claim follows. □

Continuity follows. Recall other parts of the proof are still left to the reader (well-defined, inverse). □

Definition. For a finitely generated group $\Gamma = \langle S \rangle$, we define $\text{Ends}(\Gamma)$ to be $\text{Ends}(\Delta(\Gamma, S))$ for any finite generating set S. (We need finite to know that Δ is a proper space.)

Example. Let $\Gamma_1 = \mathbb{Z}^n$ for $n \geq 2$. Then Γ_1 has one end. Let $\Gamma_2 = F_n$ for $n \geq 2$. Then $|\text{Ends}(\Gamma_2)| = \infty$. So $\mathbb{Z}_n \not\sim_{QI} F_n$ for $n \geq 2$.

Definition. A *right-angled Coxeter group* is a group with a presentation

$$\langle a_1, a_2, \ldots, a_n \mid a_i^2, [a_i, a_j] \text{ for some } i, j\rangle.$$

That is, the group is finitely generated by elements of order 2, and any additional relations are given by commutators. We often right RACG for right-angled Coxeter group.

We can represent a RACG by a graph as follows. We have one vertex for each generator, and we connect two generators by an edge if their commutator is a relation.

Fact. If the graph is disconnected, then the group can be written as a free product. Any time there is a separating subgraph, we can write the group as an amalgamated free product.

Definition. If v is a vertex in a graph, then the *link* of v, denoted $\text{Lk}(v)$, is the 1-neighborhood of v. That is, $\text{Lk}(v)$ is the subgraph consisting of v and all of the neighbors of v.

Theorem. *A RACG is infinite ended if and only if there exists a complete separating subgraph K such that there exists a vertex $v \in \Gamma \setminus K$ such that $\text{Lk}(v)$ does not contain K.*

Definition. Let X be a proper geodesic metric space. A *quasi-geodesic in X* is a quasi-isometric embedding $\varphi \colon I \to X$ of an interval (or, sometimes, I is the intersection of an interval with $\mathbb{Z}$). Note: that I may be a ray or all of $\mathbb{E}^1$. In the former case, φ is called a *quasi-ray*, and in the latter case, φ is called a *quasi-line.*

Fact. Let $c \colon [0, \infty) \to \mathbb{E}^2$ be given by $c(t) = (t, \log(1 + t)) = (r, \theta)$ in polar coordinates. (This is the logarithmic spiral going counter-clockwise outwards from the origin.) This is a quasi-ray, which is scary because it looks nothing like a geodesic.

Fact. The image of a geodesic under a QI is a quasi-geodesic.

Definition. Let (X, d) be a proper geodesic metric space. If $a, b, c \in X$, then a *geodesic triangle with vertices a, b, c* is a choice of geodesics $[a, b]$, $[b, c]$ and $[a, c]$.

Definition. Let $\delta \geq 0$. A geodesic triangle in X is *δ-slim* if each side of the triangle is contained in the δ-neighborhood of the union of the other two sides.

Definition. Let $\delta \geq 0$. X is *δ-hyperbolic* if all geodesic triangles in X are δ-slim. X is *hyperbolic* if it is δ-hyperbolic for any δ.

Example. $\mathbb{E}^2$ is not δ-hyperbolic for any δ.

Remark. If the diameter of $\{a, b, c\}$ is less than δ, then any geodesic triangle on a, b, c can be "anything". (For example, they could look "fat", or even spherical, and still be slim.)

Proposition. *X is 0-hyperbolic if and only if X is a tree.*

Example. (1) $\mathbb{E}^1$ is 0-hyperbolic.

(2) $\mathbb{H}^n$ is hyperbolic. ($\delta = \log(1 + \sqrt{2})$? Independent of n?)

10. October 7, 2010

Recall our setting from last time. X is a proper geodesic metric space. X is hyperbolic if there exists a $\delta > 0$ such that all geodesic triangles are δ-slim.

Theorem. *If X is a proper, cocompact* CAT(0) *space, then X is hyperbolic if and only if X does not contain an isometrically embedded $\mathbb{E}^2$.*

The single most important result about hyperbolic spaces is:

Theorem (Stability of quasi-geodesics)**.** *For all $\delta \geq 0$, $\lambda \geq 1$, thhere exists $R(\lambda, \delta) = R$ such that if X is δ-hyperbolic and $c\colon [0,T] \to X$ is a λ quasi-geodesic with $c(0) = p$ and $c(T) = q$, then the Hausdorff distance satisfies:*

$$d_H(\operatorname{im} c, [p,q]) < R,$$

where $[p,q]$ denotes any geodesic from p to q.

Remark. You can think of Hausdorff distance as follows: every $x \in \operatorname{im} c$ is within R of a point on $[p,q]$, and vice versa.

We may prove this later, but first we'll see why it's important. An immediate consequence:

Corollary. *Suppose X, Y are geodesic metric spaces, and let $f\colon X \to Y$ be a λ quasi-isometric embedding. If Y is δ-hyperbolic, then X is $\delta' = \delta'(\delta, \lambda)$ hyperbolic. Hence hyperbolicity is a QI invariant.*

Proof. Let a, b, c be the vertices of a triangle in X. Let c_1 be a geodesic connecting a to b; c_2 connects b to c; and c_3 connects c to a. Then $f(a), f(b), f(c)$ are connected by the images of the c_i, which are quasi-geodesics. Since Y is a geodesic space, we can draw a geodesic triangle between these vertices as well.

Take $p \in c_1$. By the theorem, there is some point z on $f(c_1)$ within R of $f(p)$. Since the geodesic triangle in Y is δ-slim, there is some point z', for example, on $f(c_3)$ within δ of z. Then, by the theorem again, there is a point $f(q)$ on $f(c_3)$ within R of z'. By the quasi-isometric embedding inequality:

$$d(p,q) < \lambda(2R + \delta) + \lambda^2.$$

Now, taking $\delta' = \lambda(2R + \delta) + \lambda^2$, the geodesic triangle in X is δ'-slim. □

Corollary. *If $X \sim_{QI} Y$, then X is hyperbolic if and only if Y is hyperbolic.*

Definition. A finitely generated group is *hyperbolic* (or *Gromov hyperbolic*, or *word hyperbolic*) if any Cayley graph of Γ is hyperbolic.

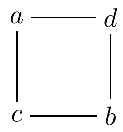
Example. Free groups are hyperbolic, since trees are 0-hyperbolic. $\Gamma = \pi_1(S_g)$ for $g \geq 2$.

Remark. In general, it's difficult to tell if the Cayley graph is hyperbolic. The hard part is really understanding the algebra of geodesics. To show a given group Γ is hyperbolic, we hope to find a nice hyperbolic metric space on which Γ acts geometrically. Then we apply Śvarc–Milnor.

Example. Consider $\mathbb{Z} \oplus \mathbb{Z}$. Geodesics from $(0,0)$ to (p,q) can be arbitrarily far apart (as p and q get large). So $\mathbb{Z} \oplus \mathbb{Z}$ is not hyperbolic. This is an example where we fully understand the geodesics in the graph.

Theorem. *If Γ is hyperbolic, then Γ contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup.*

Remark. In the graph representing a RACG, a copy of $\mathbb{Z} \oplus \mathbb{Z}$ would be represented by a non-chordal cycle of length 4:

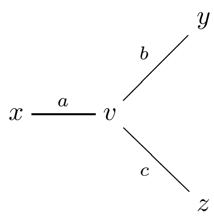


Theorem (Moussong)**.** *If you don't see the above in the defining graph, then the group is hyperbolic.*

If Γ is any finitely presented group that does not contain a $\mathbb{Z} \oplus \mathbb{Z}$, then is Γ hyperbolic? No. Consider $B(m,n) = \langle a, b \mid b^{-1}a^m b = a^n \rangle$ for $m \neq n \geq 1$. What if Γ does not contain $\mathbb{Z} \oplus \mathbb{Z}$ or $B(m,n)$? No. The examples are much harder, by work of N. Brady (1999 or 2000).

Open Question. *Suppose Γ admits a finite $K(\Gamma, 1)$ and does not contain $B(m,n)$ for $m, n \geq 1$. Then is Γ hyperbolic?*

We'll discuss various other formulations of hyperbolicity. For example, hyperbolic Cayley graphs behave a lot like trees. Let x, y, z be a triangle in X. There exist $a, b, c \geq 0$ such that $d(x,y) = a + b$, $d(x,z) = a + c$, and $d(y,z) = b + c$. Now we construct the graph:



The map which sends the triangle to this graph, we will denote by $\chi_\triangle$. We can determine the "thinness" of the triangle x, y, z by looking at the diameter $\operatorname{diam} \chi_\triangle^{-1}(v)$.

Definition. The triangle x, y, z is *δ-thin* if for all $t \in T(a,b,c)$, $\operatorname{diam} \chi_\triangle^{-1}(t) \leq \delta$. We call $\operatorname{diam} \chi_\triangle^{-1}(v)$ the *insize* of the triangle.

Proposition. *The following are equivalent:*

1. *There exist δ_0 such that all geodesic triangles in X are δ_0-slim.*
2. *There exist δ_1 such that all geodesic triangles in X are δ_1-thin.*
3. *There exist δ_2 such that the insize of any geodesic triangle is at most δ_2.*

Suppose $\Gamma = \langle A \mid R \rangle$ is a finite presentation.

Problem 1 (Word problem)**.** Can it be decided in a finite number of steps whether a given word $w \in \{A \cup \overline{A}\}^*$ has $\varepsilon(w) = e_\Gamma$?

Suppose you could find a set of elements $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$ a set of words such that

1. $\varepsilon(u_i) = \varepsilon(v_i)$,
2. $|v_i| < |u_i|$, and
3. if a word $w \in \{A \cup \overline{A}\}^*$ has $\varepsilon(w) = e_\Gamma$, then w contains at least one u_i as a subword.

If the answer is yes, then we have a very simple algorithm: look through w and check for one of the u_i. If found, replace with v_i. Since this reduces the length, applying this check recursively yields an algorithm that is linear in the length of w.

Definition. If we have a finite list of words as above, we say $\langle A \mid R\rangle$ is a *Dehn presentation* for Γ.

Theorem. *If Γ is hyperbolic, then Γ has a Dehn presentation.*

Remark. In fact, this is if and only if.

11. October 12, 2010

Proposition. *Suppose $c, c' \colon [a,b] \to X$ are two geodesics from x to y. If X is a δ-hyperbolic geodesic space, then $\operatorname{im}(c) \subset N_\delta(\operatorname{im}(c'))$ and $\operatorname{im}(c') \subset N_\delta(\operatorname{im}(c))$, where $N_\delta(U)$ denotes the δ-neighborhood of a set U.*

Proof. If $d(x,y) = \ell(c) = \ell(c') < 2\delta$, then the claim is trivial. So suppose $d(x,y) > 2\delta$. Let $z \in \operatorname{im}(c)$ such that $d(x,z), d(z,y) > \delta$. Let q be the point distance δ along the geodesic towards y. Now we have a triangle $\{q,x,y\}$ which is δ-slim by hyperbolicity. So either $d(z, \operatorname{im}(c')) < \delta$, or $d(z,q') < \delta$ for q' along the geodesic from q to y. But the latter is impossible by construction, hence $d(z,\operatorname{im}(c')) < \delta$. □

Definition. Let X be a geodesic space. Fix $k > 0$. A path $c\colon [a,b] \to X$ in X is called a *k-local geodesic* if $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [a,b]$ with $|t-t'| \le k$.

Theorem. *Let X be a δ-hyperbolic geodesic space. Fix $k > 8\delta$. Suppose $c\colon [a,b] \to X$ is a k-local geodesic.*

1. $\operatorname{im}(c) \subset N_{2\delta}([c(a), c(b)])$. *(Recall the notation $[c(a), c(b)]$ denotes any geodesic from $c(a)$ to $c(b)$.)*
2. $[c(a), c(b)] \subset N_{3\delta}(\operatorname{im}(c))$.
3. *c is a λ-quasi-geodesic for $\lambda = \lambda(k,\delta)$.*

Proof. (1) Fix a geodesic $[c(a), c(b)]$, which we will denote $[c(a), c(b)]$. If $b - a < 8\delta$, then c is a geodesic, so that $\operatorname{im}(c) \subset N_\delta([c(a), c(b)])$.

Let $t \in [a,b]$ such that $d(c(t), [c(a), c(b)])$ is maximal, and denote $c(t) = x$. Choose a subinterval $[a', t]$ of $[a,b]$ with length at most $k/2$ and at least 4δ, and let $y = c(a')$. Choose an analgous subinterval $[t, b']$ and take $z = c(b')$. Draw geodesics from y and z to $[c(a), c(b)]$, and denote the images by y' and z', respectively.

Now the restriction of c to $[a', b']$ is a geodesic, so we have a geodesic quadrilateral $\{y, z, z', y'\}$. Draw a diagonal from y' to z to get two geodesic triangles. There exists a point $w \in [y,y'] \cup [y', z'] \cup [z', z]$ such that $d(w,x) \le 2\delta$. We want $w \in [y', z']$ to complete the proof of the first claim.

Suppose $w \in [y, y']$. Then we claim there would be a path from x to y' via w which is shorter than $[y, y']$. We have

$$\begin{aligned} d(x,y') - d(y,y') &\le (d(x,w) + d(w,y')) - (d(y,w) + d(w,y')) \\ &= d(x,w) - d(y,w) \\ &\le d(x,w) - (d(y,x) - d(x,w)) \\ &= 2d(x,w) - d(y,x) \\ &< 4\delta - 4\delta. \end{aligned}$$

That is, $d(x, y') < d(y, y')$, contradicting our choice of t. Hence $w \notin [y, y']$. An analogous argument shows $w \notin [z, z']$, hence $w \in [y', z']$ and we are done.

(2) Pick an arbitrary point $p \in [c(a), c(b)]$. Any point of $\operatorname{im}(c)$ lies in either $N_{2\delta}([c(a), p])$ or $N_{2\delta}([p, c(b)])$. There exists some $x \in \operatorname{im}(c)$ in both neighborhoods, since $\operatorname{im}(c)$ is connected. Let $q \in [c(a), p]$ and $r \in [p, c(b)]$ such that $d(q, x) < 2\delta$ and $d(x, r) < 2\delta$. Now $\{q, r, x\}$ forms a δ-slim triangle. Without loss of generality, p is within δ of $[q, x]$, hence $d(p, x) \leq 3\delta$, and we are done.

(3) We will not prove the third claim today.

□

Corollary. *Let X be a δ-hyperbolic geodesic space, and let k be a fixed positive constant greater than 8δ. Let $c\colon [a, b] \to X$ be a k-local geodesic. Then either c is a constant map at $c(a) = c(b)$, or $c(a) \neq c(b)$.*

Proof. Without loss of generality, let $a = 0$. Suppose $c(a) = c(b)$. Then $\operatorname{im}(c) \subset N_{2\delta}(c(a))$ by the theorem. If there exists $t \in [0, b]$ with $8\delta < t \leq k$, then $d(c(a), c(t)) > 8\delta$, a contradiction. This implies c is geodesic, and therefore constant. □

Theorem. *Any hyperbolic group has a Dehn presentation.*

Proof. Consider $\Delta = \Delta(\Gamma, S)$ for some δ-hyperbolic group Γ. (Suppose Δ is the δ-hyperbolic Cayley graph.) Fix $k > 8\delta$. Any edge loop of length greater than 8δ in Δ will have a subpath of length in $(8\delta, k]$ that is not geodesic. This will allow us to build a Dehn algorithm.

Let U be the set of all reduced words in $\{S \cup \overline{S}\}^*$ of length at most k. Observe that U is a finite set. We need to construct a finite list of words $u_1, \ldots, u_m$ andd $v_1, \ldots, v_m$ such that:

(1) $\varepsilon(u_i) = \varepsilon(v_i)$,
(2) $|v_i| < |u_i|$, and
(3) if w is a word with $\varepsilon(w) = e_\Gamma$, then w contains some u_i.

If $R = \{u_i v_i^{-1}\}$, then $\langle S \mid R\rangle$ is a presentation for Γ.

For each $u_i \in U$, choose v_i to be a geodesic with $\varepsilon(u_i) = \varepsilon(v_i)$. If u_i is a geodesic, we leave it out of the list. This gives a Dehn algorithm. □

Remark. Let $p \in \mathbb{E}^2$, and let c_1, c_2 be geodesics leaving p. Fix t, and connect $c_1(t)$ and $c_2(t)$ by a path outside $B(p, t)$ of length at most πt.

Now consider p, c_1, c_2 in $\mathbb{H}^2$. How long is a path from $c_1(t)$ to $c_2(t)$ that lies outside $B(p, t)$? It turns out it's exponential in t.

Definition. Let X be a geodesic space. We say $e\colon \mathbb{N} \to \mathbb{R}$ is a *divergence function for X* if for all $R, r \in \mathbb{N}$ and all geodesics $c_1\colon [0, a_1] \to X$, $c_2\colon [0, a_2] \to X$ satisfying

(1) $c_1(0) = c_2(0) =: x$,
(2) $R + r \leq \min\{a_1, a_2\}$,
(3) $d(c_1(R), c_2(R)) > e(0)$,

any path connecting $c_1(R + r)$ to $c_2(R + r)$ outside $B(x, R + r)$ has length at least $e(r)$.

12. October 14, 2010

The following are some consequences of having a Dehn presentation.

Proposition. *Suppose $\Gamma = \langle S \mid R\rangle$ has a Dehn presentation. Then Γ has only finitely many conjugacy classes of finite order elements.*

Proof. Suppose $g \in \Gamma$ has order n. Let $[g]$ denote the conjugacy class of g. Let $\langle S \mid R\rangle$ be a Dehn presentation of Γ. Let $w \in \{S \cup \overline{S}\}^*$ be shortest among all words v with $\varepsilon(v) \in [g]$. We know that $\varepsilon(w^n) = e_\Gamma$.

Because we have a Dehn presentation, there exists a relator $r \in R$ such that the path w^n contains a subword that consists of more than half of r.

Claim. $\ell(w) < \ell(r)$.

Proof. HW. □

Given the claim, take $k = \max\{\ell(r) \mid r \in R\}$. Then every element of finite order is conjugate to some element of length at most k. Since there are finitely many such elements, this completes the proof. □

Definition. Let $\Gamma = \langle S \mid R\rangle$ be a finite presentation. If w is a freely reduced word in $\{S \cup \overline{S}\}^*$ of length $\ell(w)$ and $\varepsilon(w) = e_\Gamma$, then we can write

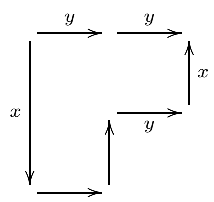
$$w = \prod_{i=1}^{N} p_i r_i^{\epsilon_i} p_i^{-1}$$

where $\epsilon = \pm 1$, and $p_i \in \{S \cup \overline{S}\}^*$. If there exists a constant k such that for all such w, $N < k\ell(w)$, then we say that this presentation satisfies a *linear isoperimetric inequality.*

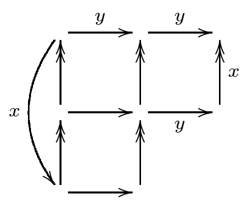
Proposition. *If Γ has a Dehn presentation $\langle S \mid R\rangle$, then this presentation satisfies a linear isoperimetric inequality with $k = 1$.*

Remark. In fact, the converse is true. A group is word hyperbolic if and only if it has a Dehn presentation if and only if it has a presentation satisfying the linear isoperimetric inequality.

Example. Let $\Gamma = \langle x, y \mid r_1, r_2\rangle \cong \mathbb{Z} \times \mathbb{Z}_3$, where $r_1 = x^3$ and $r_2 = xyx^{-1}y^{-1}$. Let $w = y^2x^{-1}y^{-1}x^{-1}y^{-1}x^{-1}$. Then we can draw w as



We can fill in this picture in the Cayley graph:



The three squares are r_2, and the half circle is r_1. So

$$w = (x^{-1}r_2x)(x^{-1}yr_2y^{-1}x)(x^{-2}r_2x^2)(r_1^{-1}).$$

Definition. Let X be a meteric space. A *path* is a continuous map $c\colon [a,b] \to X$. The *length* of the path is

$$\ell(c) = \sup_{a=t_0\leq\cdots\leq t_n=b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})).$$

Note that this could be infinite. If it is finite, we say that c is *rectifiable.*

Fact. If $c\colon [a,b] \to X$ is rectifiable with $\ell(c) = \ell$, then we can reparametrize c as $\widetilde{c}\colon [0,\ell] \to X$ in such a way that $\ell(\widetilde{c}|_{[0,t]}) = t$.

Remark. We really want to think of this reparametrization as $[0,1] \xrightarrow{\lambda} [0,\ell] \xrightarrow{\widetilde{c}} X$, where λ is linear. By an abuse of notation, we will call the composition c. This composition should be parametrized proportional to arc length.

Proposition. *Let X be a δ-hyperbolic geodesic space, let $c\colon [0,1] \to X$ be a rectifiable path parametrized proportional to arc length, and take $p = c(0)$, $q = c(1)$. If $[p,q]$ is a geodesic between p and q in X, then for all $x \in [p,q]$,*

$$d(x, \operatorname{im}(c)) \leq \delta|\log_2 \ell(c)| + 1.$$

Proof. If $\ell(c) \leq 1$, then this is trivial: if $\ell(c) \leq 1$, then $d(p,q) \leq 1$, and the right hand side of the inequality can never be less than 1.

Suppose $\ell(c) > 1$. Let N be the integer such that $\ell(c)/2^{N+1} < 1 \leq \ell(c)/2^N$. Consider the δ-thin triangle $\{p, c(1/2), q\}$. There exists $y_1 \in [p, c(1/2)] \cup [c(1/2), q]$ such that $d(x, y_1) < \delta$. Without loss of generality, suppose $y_1 \in [p, c(1/2)]$. Now draw the δ-thin triangle $\{p, c(1/4), c(1/2)]$. Then, without loss of generality, there is some $y_2 \in [c(1/4), c(1/2)]$ such that $d(y_1, y_2) < \delta$.

Proceed by induction so that at the $n+1$ stage, we have $y_n \in [c(t_n), c(t'_n)]$. To get y_{n+1}, we build triangle $\triangle_{n+1}$ as $\{c(t_n), c(t'_n), c((t_n + t'_n)/2)\}$. Then $d(y_n, y_{n+1}) < \delta$.

At the Nth stage, we have y_N with $d(y_N, x) \leq \delta N$. Now y_N is along a geodesic from $c(t_N)$ to $c(t'_N)$. Suppose without loss of generality that y_N is closest to $c(t_N)$. Then

$$d(x, c(t_N)) \leq d(x, y_N) + d(y_N, c(t_N)) \leq \delta N + \ell(c)/2^{N+1} \leq \delta|\log_2 \ell(c)| + 1.$$

□

Proposition. *If X is a δ-hyperbolic geodesic space, then X has an exponential convergence function.*

Proof. Let

$$e(r) = \max\{3\delta, 2^{\frac{r-1}{\delta}}\}.$$

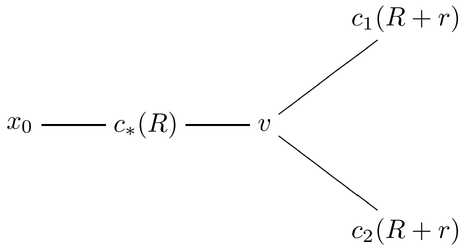
Suppose $d(c_1(R), c_2(R)) > 3\delta$ for geodesics c_1, c_2 starting at some point x_0. Consider the δ-thin triangle $\{x_0, c_1(R+r), c_2(R+r)\}$. From the definition of δ-thin, we can find $i_1 \in [x_0, c_1(R+r)]$, $i_2 \in [x_0, c_2(R+r)]$, and $i_3 \in [c_1(R+r), c_2(R+r)]$ such that

$$\begin{aligned} d(x_0, c_1(R+r)) &= d(x_0, i_1) + d(i_1, c_1(R+r)) \\ d(x_0, c_2(R+r)) &= d(x_0, i_2) + d(i_2, c_2(R+r)) \\ d(c_1(R+r), c_2(R+r)) &= d(c_1(R+r), i_3) + d(i_3, c_2(R+r)) \end{aligned}$$

Also note that, in particular, $d(i_k, i_j) \le \delta$ for $i, j \in \{1, 2, 3\}$.

Claim. $d(x_0, i_1) < d(x_0, c_1(R))$ *(and similarly for* i_2 *and* c_2*).*

Proof. Otherwise, when we collapse the triangle to a tripod, we have:



where the preimage of v under the collapsing map is $\{i_1, i_2, i_3\}$ and the preimage of $c_*(R)$ is $\{c_1(R), c_2(R)\}$. But then by the definition of δ-thin, we must have that the preimage of $c_*(R)$ has diameter at most δ, contradicting the assumption that $d(c_1(R), c_2(R)) \ge 3\delta$. □

Claim. $d(i_1, x_0) = d(i_2, x_0) < R - \delta$.

Proof. Now suppose $d(x_0, i_1) = d(x_0, i_2) \ge R - \delta$. We have:

$$\begin{aligned} d(x_0, c_1(R)) &= d(x_0, i_1) + d(i_1, c_1(R)) \\ &\Rightarrow R \ge R - \delta + d(i_1, c_1(R)) \\ &\Rightarrow \delta \ge d(i_1, c_1(R)). \end{aligned}$$

Similarly, $\delta \ge d(i_2, c_2(R))$. By the triangle inequality:

$$\begin{aligned} d(c_1(R), c_2(R)) &\le \underbrace{d(c_1(R), i_1)}_{\le \delta} + \underbrace{d(i_1, i_2)}_{< \delta} + \underbrace{d(i_2, c_2(R))}_{\le \delta} \\ &< 3\delta, \end{aligned}$$

contradicting the assumption that $d(c_1(R), c_2(R)) \ge 3\delta$. This completes the proof of the claim. □

Claim. $d(x_0, i_3) < R$.

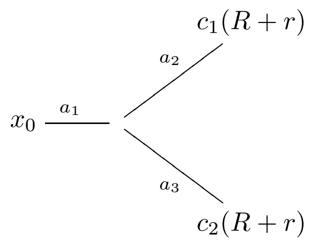
Proof. By the triangle inequality: $d(x_0, i_3) \le d(i_3, i_1) + d(i_1, x_0)$. □

Now we have $B(i_3, r) \subset B(x_0, R + r)$. Let γ be a rectifiable path outside of $B(x_0, R + r)$ from $c_1(R + r)$ to $c_2(R + r)$. (If the path is not rectifiable, the conclusion is trivial.) By the proposition,

$$r < d(i_3, \mathrm{im}(\gamma)) \le \delta |\log_2 \ell(\gamma)| + 1.$$

So $(r-1)/\delta \le |\log_2 \ell(\gamma)|$. If $\ell(\gamma) \ge 1$, then by exponentiating we have $2^{\frac{r-1}{\delta}} \le \ell(\gamma)$. It remains to show that $\ell(\gamma) \ge 1$.

The triangle $\{x_0, c_1(R+r), c_2(R+r)\}$ collapses to the following tripod:



Note that since c_1 and c_2 are geodesics, we have $a_2 = a_3$. Furthermore, by the first claim, we know that the image of $c_1(R)$ under the collapsing map must lie along the edge marked a_2, and the image of $c_2(R)$ must lie along the edge marked a_3. Since $d(c_1(R), c_1(R+r)) = r$, and similarly for c_2, we have that $a_2 = a_3 > r$. It follows that $d(c_1(R+r), c_2(R+r)) \geq 2r > 1$, hence $\ell(\gamma) > 1$, completing the proof. □

13. October 19, 2010

From last time, you can show a group is not hyperbolic by showing that the Dehn function is not linear. But this isn't much easier than showing that the Cayley graph is not hyperbolic.

Proposition. *Suppose Γ is a hyperbolic group. Let $\gamma \in \Gamma$ have infinite order, $\Delta = \Delta(\Gamma, S)$ be any Cayley graph, and α a geodesic from e_Γ to γ. Then the bi-infinite path*

$$(\dots, \gamma^{-1}\alpha, \alpha, \gamma\alpha, \gamma^2\alpha, \dots)$$

is a quasi-geodesic line. As a consequence, the map $\mathbb{Z} \to \gamma$ given by $n \mapsto \gamma^n$ is a QI embedding.

Example. Consider $\Gamma = BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$. We claim that $\langle b \rangle$ does not determine a quasi-geodesic line in $\Delta(\Gamma, S)$ for $S = \{a, b\}$.

First notice that $a^n ba^{-n} = b^{2^n}$. In the subgroup $\langle b \rangle$, the element b^{2^n} has length 2^n. But in the full group, it has length at most $2n+1$, so $\langle b \rangle$ does not determine a quasi-geodesic line. It follows that $BS(1,2)$ is not a hyperbolic group.

Fact. If Γ is hyperbolic and $\gamma \in \Gamma$ has infinite order, then $|p| = |q|$ when γ^p is conjugate to γ^q.

Proof. Proof by picture (draw the brick wall). Finish on your own. (Hint: q^{m-1} copies of γ^q are conjugate via t to q^{m-1} copies of γ^p.) □

Conjecture. *If Γ has a finite $K(\pi, 1)$ and does not contain $B(p,q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$, then Γ is hyperbolic.*

Definition. Recall that a $K(\Gamma, 1)$ is a CW complex with fundamental group isomorphic to Γ, and contractible universal cover.

Fact. A finitely generated Abelian group is hyperbolic if and only if it contains a cyclic subgroup of finite index.

Proof. If Γ contains a cyclic subgroup of finite index, then it is either finite or 2-ended. In either case, it is hyperbolic.

Conversely, if Γ is finitely generated and Abelian, then by the structure theorem either Γ is finite or virtually $\mathbb{Z}^n$. The former case is trivial. In the latter case, $\Gamma \sim_{QI} \mathbb{E}^n$, which is hyperbolic only when $n = 1$. So Γ is virtually $\mathbb{Z}$. □

Theorem (N. Brady)**.** *Not every finitely presented subgroup of a hyperbolic group is hyperbolic.*

Remark. However, every subgroup of a free group is free.

Fact. If N is a finitely generated, nontrivial, normal subgroup of a finitely generated free group F, then $[F : N] < \infty$. (Original proof due to Greenberg.)

Definition. Let X be a geodesic space. A subset $C \subset X$ is called *quasi-convex* if there exists a $k > 0$ such that for all $x, y \in C$, each geodesic satisfies $[x, y] \subset N_k(C)$.

Lemma. *Suppose* $\Gamma = \langle A \rangle$ *is any finitely generated group and* $H \leq \Gamma$*. If* H *is quasi-convex in* $\Delta = \Delta(\Gamma, A)$*, then* H *is finitely generated and* $H \hookrightarrow \Gamma$ *is a QI-embedding.*

Proof. Let $h \in H$, and suppose $h = a_1 \cdots a_n$ is a geodesic for $h \in \Delta(\Gamma, A)$. By quasi-convexity, this geodesic is in $N_k(H)$. For each i, choose geodesics u_i of length at most k connecting a_i to $h_i \in H$. Then $h_i = u_{i-1} h_i u_i^{-1}$. Then we can write $h = h_1 \ldots, h_n$, and for each i, $|h_i|_A \leq 2k + 1$. Taking $S = \overline{B(e_\Gamma, 2k + 1)}$ in $\Delta \cap H$, we have a finite generating set for H.

From the above, we have $d_S(e_\Gamma, h) \leq d_A(e_\Gamma, h)$, since for any path of length n in the A metric, we have constructed a path of length n in the S metric. On the other hand, if $d_A(e_\Gamma, h) \leq 2k + 1$, then $d_S(e_\Gamma, h) = 1$ by choice of S. So

$$d_S(e_\Gamma, h) \leq d_A(e_\Gamma, h) \leq (2k + 1) d_S(e_\Gamma, h).$$

This shows that $(H, S) \to (\Gamma, A)$ is a QI embedding. □

Proposition. *If* (Γ, A) *is hyperbolic and* (H, B) *is a finitely generated subgroup, then* H *is quasi-convex if and only if* H *is QI embedded in* Γ*.*

Corollary. *If* H *is quasi-convex in* (Γ, S)*, then* H *is quasi-convex in* (Γ, T) *for any finite generating sets* S, T*.*

14. October 21, 2010

Recall the lemma from last time:

Lemma. *Let* (Γ, A) *be a finitely generated group and* $H \leq \Gamma$ *a subgroup. If* H *is quasi-convex in* $\Delta(\Gamma, A)$*, then* H *is finitely generated and* $H \hookrightarrow \Gamma$ *is a QI embedding.*

Corollary. *If* (Γ, A) *is hyperbolic with finite generating set* A *and* (H, B) *is a finitely generated subgroup of* Γ*, then:*

(1) *if* H *is quasi-convex with respect to one finite generating set, then it is quasi-convex with respect to all finite generating sets, so that we can say* H *is quasi-convex in* Γ*; and*
(2) H *is quasi-convex in* Γ *if and only if* H *is QI embedded in* Γ *via the inclusion map.*

Proof. We will prove the second claim first. By the lemma, if H is quasi-convex then it is QI embedded. Conversely, suppose H is QI embedded. We have an induced inclusion map $\Delta_H = \Delta(H, B) \hookrightarrow i\Delta_\Gamma = \Delta(\Gamma, A)$. Let $[h_1, h_2]_H$ be a geodesic in Δ_H. Then, since the inclusion is a QI embedding, the image $[h_1, h_2]_\Gamma$ is a quasi-geodesic in Δ_Γ. From the stability of quasi-geodesics, we know there exists $R > 0$ such that $[h_1, h_2]_\Gamma \subset N_R(i([h_1, h_2]_H))$. Taking $k = R + \frac{\lambda}{2}$, we have $[h_1, h_2]_\Gamma \subset N_k(H)$, and H is quasi-convex.

The first claim follows from the second, since all Cayley graphs are QI to each other. □

Proposition. *Suppose Γ is hyperbolic. If $H \leq \Gamma$ is quasi-convex, then H is finitely generated and QI embedded. Furthermore, H is hyperbolic.*

Proof. It only remains to show that H is hyperbolic. We have seen previously that if $X \to X'$ is a QI embedding and X' is hyperbolic, then X is hyperbolic. Taking $H = X$ and $\Gamma = X'$, H is hyperbolic. □

Exercise. Every finitely generated subgroup of a finitely generated free group is quasi-convex. (For HW.)

Exercise (Much harder, but fun)**.** Every finitely generated subgroup of a hyperbolic surface group (that is, $\pi_1(S_g)$ for $g \geq 2$) is quasi-convex.

Remark. This is not true in general for hyperbolic groups. In fact, there exist a hyperbolic group Γ and finitely presented subgroup H such that H is not hyperbolic. (N. Brady)

Example (Cannon–Thurston)**.** It is also possible to have hyperbolic subgroups of a hyperbolic group that are not quasi-convex (that is, that are not QI embedded). Consider $\Gamma = \pi_1(M^3)$ for a compact, hyperbolic 3-manifold M^3. Take $H = \pi_1(S_g)$ to be a surface subgroup with $g \geq 2$. In fact, H is normal.

Lemma. *If H_1, H_2 are quasi-convex in Γ, then $H_1 \cap H_2$ is quasi-convex in Γ.*

Proof. Left to the reader. □

Proposition. *If Γ is hyperbolic and $\gamma \in \Gamma$, then $C_\Gamma(\gamma)$ is quasi-convex.*

Corollary. *If Γ is hyperbolic and $\gamma \in \Gamma$ has infinite order, then $\mathbb{Z} \to \Gamma$ given by $n \mapsto \gamma^n$ is a QI embedding.*

Proof. It is sufficient to show that $\langle \gamma \rangle$ is quasi-convex in Γ. Suppose the order of γ is infinite. $C_\Gamma(\gamma)$ is quasi-convex by the proposition, and therefore finitely generated by, say, S. Now

$$Z(C_\Gamma(S)) = \bigcap_{s \in S} C_\Gamma(s)$$

is finitely generated, Abelian, and hyperbolic. So $Z(C_\Gamma(\gamma))$ is virtually cyclic. Since $\langle \gamma \rangle \subset Z(C_\Gamma(\gamma))$, so it is finite index in this center. (In fact, $\langle \gamma \rangle$ and $Z(C_\Gamma(\gamma))$ are both 2-ended, and by the inclusion, they are QI.) Therefore

$$\langle \gamma \rangle \to Z(C_\Gamma(\gamma)) \to C(\gamma) \to \Gamma$$

is a chain of QI embeddings, and is therefore a QI embedding. □

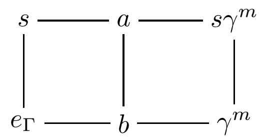
Corollary. *Let Γ be hyperbolic and $\gamma \in \Gamma$ have infinite order. There exists a constant L such that for all x on a geodesic $[\gamma^i, \gamma^j]$ there exists k such that $d(x, \gamma^k) < L$.*

Proof. Let α be a geodesic from e_Γ to γ. Connect γ^i to γ^j by a quasi-geodesic by taking geodesics $\gamma^i \cdot \alpha, \gamma^{i+1} \cdot \alpha$, and so on. Fix a geodesic $[\gamma^i, \gamma^j]$ and take x on the geodesic. Project x to a point on the quasi-geodesic at distance at most R (from the stability of quasi-geodesics). This image is between two consecutive powers of γ. Take $L = R + \ell(\gamma)/2$. □

Corollary. *If* Γ *is hyperbolic and* $\gamma \in \Gamma$ *has infinite order, then* $|C_\Gamma(\gamma)/\langle\gamma\rangle|$ *is finite. In particular, this shows that there are no* $\mathbb{Z} \oplus \mathbb{Z}$ *subgroups in* $C_\Gamma(\gamma)$*, hence* $C_\Gamma(\gamma)$ *is 2-ended.*

Proof. Let $s \in C_\Gamma(\gamma)$. It suffices to show that every coset $s\langle\gamma\rangle$ intersects $\overline{B(e_\Gamma, rL + 2\delta)}$, where the L comes from the previous corollary.

Let $s \in C_\Gamma(\gamma)$. Choose m such that $d(e_\Gamma, \gamma^m) > 2\ell(s) + 2\delta$. (We can do this because γ is infinite order. We have the geodesic quadrilateral:



We claim that $d(a, b)$ is less than 2δ. If there exists $p \in [e_\Gamma, s]$ and $q \in [\gamma^m, s\gamma^m]$ with $d(p, q) < 2\delta$, then $d(e_\gamma, \gamma^m) < 2\delta + 2\ell(s)$, a contradiction. So we can choose a, b as in the picture.

By the previous corollar, there are some i, j such that $d(a, s\gamma^j) < L$ and $d(b, \gamma^i) < L$. Now $d(e_\Gamma, s\gamma^{j-i}) = d(\gamma^i, s\gamma^j) \leq 2L + 2\delta$. (The first equality follows from the fact that γ^i acts like an isometry.) This completes the proof. ∎

15. October 26-28, 2010

Remark. Let $g \in \Gamma$, and let φ_g be the automorphism given by conjugation by g. Then $C_\Gamma(g) = \mathrm{Fix}(\varphi_g)$. By the following theorem, centralizers in hyperbolic groups are quasi-convex.

Theorem (Neumann). *If* Γ *is hyperbolic and* $\varphi \in \mathrm{Aut}(\Gamma)$*, then* $\mathrm{Fix}(\varphi)$ *is quasi-convex.*

Theorem (Gersten, 70s). *If* F *is a finitely generated free group and* $\varphi \in \mathrm{Aut}(F)$*, then* $\mathrm{Fix}(\varphi)$ *is finitely generated.*

Proof. Long and technical combinatorial group theory. ∎

Remark. Daryl Cooper gave a GGT proof of this last result that is much nicer.

Theorem. *Let* Γ *be hyperbolic, let* $g \in \Gamma$ *have infinite order, and let* $\alpha = [e, g]$ *be a choice of geodesic in a Cayley graph* Δ*. Define* $\beta = (\dots, g^{-1}\alpha, \alpha, g\alpha, \dots)$ *to be a bi-infinite path. Then* β *is a quasi-geodesic.*

Proof. We want to show:

$$\frac{1}{\lambda} d_\beta(x, y) - \lambda \leq d_\Delta(x, y) \leq \lambda d_\beta(x, y) + \lambda.$$

We have $d_\Delta(x, y) \leq d_\beta(x, y)$ trivially. It remains to find λ such that $d_\beta(x, y) \leq \lambda d_\Delta(x, y) + \lambda$.

Claim. *Let* $R \in \mathbb{Z}^+$*. Let* $k \in \mathbb{Z}$ *such that* $d(e, g^k) > 8R + 2\delta$*. Fix a geodesic from* e *to* g^k*, and let* y *be the midpoint of this geodesic. Let* I *be the open subsegment of this geodesic of length* $2R$ *centered at* y*. Then for any* $p \in B_R(e)$ *and* $q \in B_R(g^k)$*, the midpoint* m_1 *of* $[p, q]$ *lies within* 2δ *of* I*:*

$$d(m_1, I) < 2\delta.$$

Proof. In the following, consult Figure 1 for a visual aid. We have a 2δ-thin geodesic rectangle $\{e, p, q, g^k\}$. Fix a diagonal $[p, g^k]$ to form two δ-thin triangles $\{p, q, g^k\}$ and $\{e, p, g^k\}$. If we draw the internal points $a \in [q, g^k], b \in [p, q], c \in [p, g^k]$ for the triangle $\{p, q, g^k\}$, then we have $d(a, b) < \delta$ and $d(a, c) < \delta$. So $b, c \in B_{R+\delta}(g^k)$. Similarly, the internal points for the triangle $\{e, p, g^k\}$ lie in $B_{R+\delta}(e)$.

Let m_1 be the midpoint of $[p, q]$. Observe that $d(p, m_1) = d(q, m_1) > 3R + \delta$. Let $m_1' \in [p, g^k]$ such that $d(p, m_1') = d(p, m_1)$. Observe that, by δ-thinness, $d(m_1, m_1') < \delta$. Let m_2 be the midpoint of $[p, g^k]$. We have $d(m_2, g^k) > \frac{7}{2}R + \delta$.

We now claim that $d(m_1', m_2) < R/2$. By the reverse triangle inequality, we have $|d(p, q) - d(p, g^k)| \leq d(g^k, q) < R$. It follows that:

$$\begin{aligned} |d(p,q) - d(p,g^k)| &< R \\ \Rightarrow \big|[d(p,m_1) + d(m_1,q)] - [d(p,m_2) + d(m_2,g^k)]\big| &< R \\ \Rightarrow |2d(p,m_1) - 2d(p,m_2)| &< R \\ \Rightarrow |d(p,m_1) - d(p,m_2)| &< R/2 \\ \Rightarrow |d(p,m_1') - d(p,m_2)| &< R/2. \end{aligned}$$

Let $m_2' \in [e, g^k]$ so that $d(g^k, m_2) = d(g^k, m_2')$. By an argument analogous to the above, $d(m_2', y) < R/2$. Let $m_1'' \in [e, g^k]$ so that $d(g^k, m_1') = d(g^k, m_1'')$. Observe that $d(m_1', m_1'') < \delta$ and $d(m_2, m_2') < \delta$ by δ-thinness. Now:

$$d(m_1'', m_2') = d(m_1', m_2) < R/2.$$

By the triangle inequality, $d(m_1'', y) < R$. Now $d(m_1, I) \leq d(m_1, m_1'') < 2\delta$, completing the proof of the claim. □

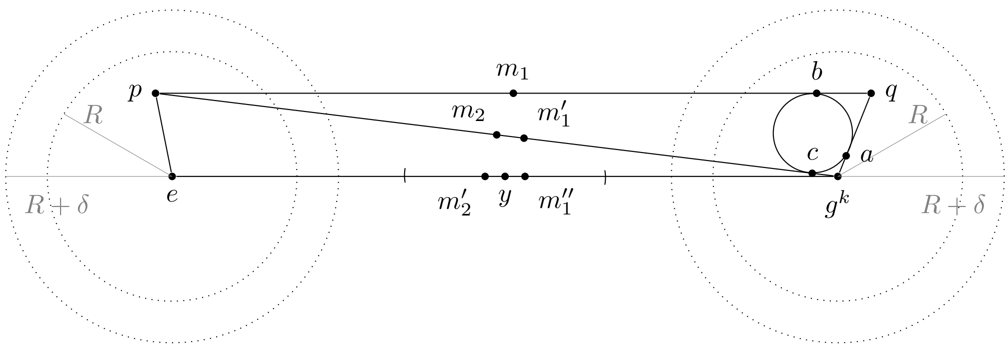


FIGURE 1. Picture to help with the proof of the first claim. This picture is not general.

Now let N be the number of vertices in $B_{2\delta}(e)$. I has at most $2R$ vertices, and each 2δ-neighborhood of a vertex contains N vertices. Therefore $N_{2\delta}(I)$ has at most $2NR$ vertices. Consider the translates of $[e, g^k]$ by the elements $e, g, \ldots, g^{2NR}$. The midpoints of these segments must all be distinct, otherwise g would fix a point (which is impossible since g has infinite order). Since there are $2NR + 1$ such midpoints, not all of them can be in $N_{2\delta}(I)$. That is, there exists some power $p(R) \leq 2NR$ such that $g^{p(R)} \notin B_R(e)$.

Observe that $d(e, g^{p(R)}) > R$, and $d_\beta(e, g^{p(R)}) = |g|p(R)$. Hence $p(R) > R/|g|$.

Claim. *For all $R \in \mathbb{Z}$, $d(e, g^{2NR}) \geq R$.*

Proof. The claim is trivial for $R \leq 0$. Suppose the claim is false. Then there exists some $R_0 > 0$ such that $d(e, g^{2NR_0}) < R_0$. Since the inequality is strict, there exists $\epsilon > 0$ such that $d(e, g^{2NR_0} < R_0 - \epsilon$. (We will need this extra wiggle room later.) Let $n \in \mathbb{Z}^+$ be large and $0 \leq R_1 < R_0$, and define $s = n(2NR_0) + R_1$. We have

$$\begin{aligned} d(e, g^s) &= d(e, g^{n(2NR_0)+R_1}) \\ &\leq d(e, g^{n(2NR_0)}) + d(e, g^{R_1}) \\ &\leq nd(e, g^{2NR_0}) + d(e, g^{R_1}) \\ &< n(R_0 - \epsilon) + d(e, g^{R_1}) \\ &= nR_0 - n\epsilon + d(e, g^{R_1}). \end{aligned}$$

That is, for all $n \in \mathbb{Z}^+$ and some fixed R_1 as above, $d(e, g^{n(2NR_0)+R_1}) < nR_0 - n\epsilon + d(e, g^{R_1})$. Note that $d(e, g^{R_1})$ does not depend on n, and since ϵ is fixed the term $n\epsilon$ can be made as large as we want by raising n. It follows that there is some $n_0 \gg 0$ such that for all $n \geq n_0$,

$$d(e, g^{n(2NR_0)+R_1}) < nR_0. \tag{1}$$

Take $n > n_0$ and choose $R > n|g|2NR_0$. By our previous discussion, there exists $p(R) \leq 2NR$ such that $d(e, g^{p(R)}) > R > nR_0$. But we have $p(R) > R/|g| > n(2NR_0)$, so $d(e, g^{p(R)}) < nR_0$ by Equation (1), a contradiction. This completes the proof of the claim. □

We will now show that β is a quasi-geodesic. Pick $x, y \in \beta$. Recall that it remains to find λ such that $d_\beta(x, y) \leq \lambda d(x, y) + \lambda$.

There exists $a \in \mathbb{Z}$ such that $d(x, g^{2aN}) \leq N|g|$. Similarly, there is some $b \in \mathbb{Z}$ so that $d(y, g^{2bN}) \leq N|g|$. Observe that, by the second claim, $d(g^{2aN}, g^{2bN}) = d(e, g^{2N|b-a|}) \geq 2|b - a|$. We have

$$\begin{aligned} d_\beta(x, y) &\leq d(x, g^{2aN}) + d(g^{2aN}, g^{2bN}) + d(g^{2bN}, y) \\ &\leq N|g| + 2N|b - a||g| + N|g| \\ &= 2N|b - a||g| + 2N|g|. \end{aligned}$$

We also have

$$\begin{aligned} 2|b - a| &\leq d(g^{2aN}, g^{2bN}) \\ &\leq d(g^{2aN}, x) + d(x, y) + d(y, g^{2bN}) \\ &\leq N|g| + d(x, y) + N|g| \\ &= 2N|g| + d(x, y). \end{aligned}$$

Now

$$\begin{aligned} d_\beta(x, y) &\leq 2N|b - a||g| + 2N|g| && \text{(first calculation)} \\ &= N|g|(2|b - a|) + 2N|g| \\ &\leq N|g|\,[2N|g| + d(x, y)] + 2N|g| && \text{(second calculation)} \\ &= 2N^2|g|^2 + N|g|d(x, y) + 2N|g| \\ &= N|g|d(x, y) + (2N^2|g|^2 + 2N|g|) \end{aligned}$$

Choosing $\lambda = 2N^2|g|^2 + 2N|g|$ completes the proof. □

Corollary. *If γ has infinite order, $\langle\gamma\rangle \cong \mathbb{Z}$ is quasi-convex.*

Remark. Recall that we have already used this to show that for infinite order γ, $|C(\gamma)/\langle\gamma\rangle| < \infty$. This shows that $C(\gamma)$ is 2-ended, and it follows that $C(\gamma)$ is quasi-convex.

Remark. In the defining graph of a word hyperbolic right angled Coxeter group, if v represents some generator, the centralizer of v is the subgroup generated by $\mathrm{St}(v)$. (Note: the *star of* v is the link of v and v itself.)

Theorem. *Given any element $g \in \Gamma$ of a hyperbolic group, $C_\Gamma(g)$ is quasi-convex.*

One proof of this theorem shows that the fixed subgroup of an automorphism $\mathrm{Fix}(\varphi)$ is quasi-convex. Then $C_\Gamma(g) = \mathrm{Fix}(\varphi_g)$, where φ_g is conjugation by g. Another proof uses the following lemma, which also solves the conjugacy problem.

Proposition. *Let (Γ, S) be a hyperbolic group with finite generating set. There exists a constant $c_0 = c_0(\delta, |S|)$ such that if $a, b \in \Gamma$ are conjugate, then there exists $x \in \Gamma$ with $x^{-1}ax = b$ and $|x|_S \leq |a|_S + |b|_S + c_0$.*

16. November 2, 2010

Remark (Philosophy). Let (Γ, A) be a finitely presented group. We can also think of Γ as $\pi_1(K)$ for some complex K. Algebraically, the word problem is about recognizing a word $w \in \{A \cup \overline{A}\}^*$ has $\varepsilon(w) = e_\Gamma$. Geometrically, the world problem is about recognizing whether a given loop is homotopically trivial.

Algebraically, the conjugacy problem is about recognizing when two words $u, w \in \{A \cup \overline{A}\}^*$ have $\varepsilon(u) = g\varepsilon(w)g^{-1}$ for some $g \in \Gamma$. Geometrically, the conjugacy problem is about recognizing when two loops are homotopic.

When K has some "nice" geometry, then there are well-known geometric techniques for solving the geometric versions of these problems. ("Nice" means that K is either negatively curved or non-positively curved. For us, negatively curved means δ-hyperbolic.)

We'll see more of this philosophy on Thursday. Today's goal is the following:

Proposition. *Suppose Γ is a hyperbolic group and $g \in \Gamma$. Then $C_\Gamma(g)$ is quasi-convex.*

Remark. We already know this is true for infinite order g, since $\langle g\rangle$ is quasi-convex and $|C_\Gamma(g)/\langle g\rangle|$ is finite.

Lemma. *There exists a constant $K = K(\delta)$ such that if σ, σ' are geodesics from x to y and x' to y', respectively, (and by reparameterizing one of them, we may assume they are maps from the same interval) in $\Delta(\Gamma, S)$ that begin and end one unit apart, then $d(\sigma(t), \sigma'(t)) \leq K$ for all t.*

Proof. 1 For all t, there exists $\widetilde{w} \in [x', y']$ such that $d(\sigma(t), \widetilde{w}) \leq 2\delta + 1$. Then $\widetilde{w} = \sigma'(\widetilde{t})$ for some $\widetilde{t}$. (We choose the smallest $\widetilde{t}$ with this property.) Without loss of generality, $t \leq \widetilde{t}$ (possibly by switching the roles of σ and σ'). We claim that $\widetilde{t} - t \leq 2\delta + 2$:

$$\begin{aligned}\widetilde{t} &= d(x', \widetilde{w}) \\ &\leq d(x', x) + d(x, \sigma(t)) + d(\sigma(t), \widetilde{w}) \\ &\leq 1 + t + 2\delta + 1.\end{aligned}$$

Now

$$\begin{aligned} d(\sigma(t), \sigma'(t)) &\leq d(\sigma(t), \widetilde{w}) + d(\widetilde{w}, \sigma'(t)) \\ &\leq 2\delta + 1 + (\widetilde{t} - t) \\ &\leq 2\delta + 1 + 2\delta + 2 \\ &= 4\delta + 3. \end{aligned}$$

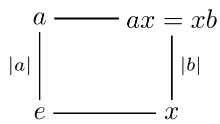
Taking $K = 4\delta + 3$, we are done. □

Lemma. *There exists a constant $K = K(\delta)$ such that if σ, σ' are geodesics from x to y and x' to y', respectively, in $\Delta(\Gamma, S)$, then $d(\sigma(t), \sigma'(t)) \leq K \max\{d(x, x'), d(y, y')\}$ for all t.*

Proof. This follows directly from the previous lemma. □

Proposition (Bridson and Howie)**.** *Let $\Gamma = \langle S \rangle$. There exists a constant c_0 (depending on δ and $|S|$) such that if $a, b \in \Gamma$ are conjugate, then there exists an $x \in \Gamma$ such that $x^{-1}ax = b$ and $|x|_S := |x| \leq |a| + |b| + c_0$.*

Proof. Since a, b are conjugate, choose some conjugating word $x \in \Gamma$ (that is, $x^{-1}ax = b$) of minimal length $|x| = n$. Let σ_x be a geodesic $[e, x]$. Then $a\sigma_x$ is a geodesic $[a, ax = xb]$. Then we have a rectangle as in the second lemma:



Each vertex $\sigma_x(i)$ along σ_x lies within 2δ of one of the other three sides of the quadrilateral.

Claim. *If i satisfies*

$$2\delta + |a| < i < n - 2\delta - |b|$$

then there exists $v \in a\sigma_x$ such that $d(\sigma_x(i), v) \leq 2\delta$.

Proof. If there exists $p \in [e, a]$ such that $d(p, \sigma_x(i)) < 2\delta$, then

$$i = d(e, \sigma_x(i)) \leq d(\sigma_x(i), p) + d(p, e) \leq 2\delta + |a|.$$

Similarly, if $p \in [x, xb]$ with $d(\sigma_x(i), p) \leq 2\delta$, then

$$n - i = d(\sigma_x(i), x) \leq d(\sigma_x(i), p) + d(p, x) \leq 2\delta + |b|.$$ □

For any i in this range, there exists a vertex $v \in [a, ax]$ such that $d(\sigma_x(i), v) \leq 2\delta$. We can write $v = a\sigma_x(j(i))$.

Claim. *$|j(i) - i| \leq 2\delta$.*

Proof. Suppose without loss of generality, $j(i) > i$. If the claim is false, then

$$\begin{aligned} |ax| &\leq d(e, \sigma_x(i)) + d(\sigma_x(i), v) + d(v, ax) \\ &\leq i + 2\delta + (n - j(i)) \\ &< n. \end{aligned}$$

Since ax also conjugates a to b, this contradicts the minimality of the length of x. □

Now we have $d(\sigma_x(i), a\sigma_x(i)) \leq 4\delta$.

Claim. *The group elements $\sigma_x(i)^{-1}a\sigma_x(i)$ are distinct for i in the appropriate range.*

Proof. Suppose not. Then $\sigma_x(i)^{-1}a\sigma_x(i) = \sigma_x(j)^{-1}a\sigma_x(j)$ for some $j > i$. Let

$$x' = \sigma_x(i)\sigma_x(j)^{-1}x \in \Gamma.$$

We will show that $ax' = x'b$ and $|x'| < |x|$, contradicting minimality of the length of x. We have

$$\begin{aligned} ax' &= a\sigma_x(i)\sigma_x(j)^{-1}x \\ &= \sigma_x(i)\sigma_x(j)^{-1}\underbrace{ax}_{=xb} \\ &= \underbrace{\sigma_x(i)\sigma_x(j)^{-1}x}_{=x'}\,b \\ &= x'b. \end{aligned}$$

Write out $x = s_1s_2\cdots s_n$, $\sigma_x(i) = s_1s_2\cdots s_i$, and $\sigma_x(j) = s_1\cdots s_j$. Then

$$\begin{aligned} x' &= (s_1\cdots s_i)(s_j^{-1}\cdots s_1^{-1})s_1\cdots s_n \\ &= s_1\cdots s_is_{j+1}\cdots s_n. \end{aligned}$$

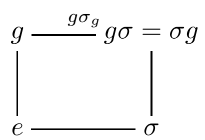
So $|x'| \le n - (j - i)$. □

Recall that the word length of each $\sigma_x(i)^{-1}a\sigma_x(i)$ is at most 4δ, so the set $\{\sigma_x(i)^{-1}a\sigma_x(i)\}$ for i in the appropriate range is in $B(4\delta, e_\Gamma)$. Hence the number of possible i's, $n - 2\delta - |b| - (2\delta + |a|)$, is at most the number of vertices in $B(4\delta, e_\Gamma)$. If V is the number of vertices in $B(4\delta, e_\Gamma)$, then

$$n \le V + |a| + |b| + 4\delta.$$

Taking $c_0 = V + 4\delta$, we are done. □

Proof of the first proposition. Let $\gamma \in C_\Gamma(g)$. We must show that there exists Q such that each vertex along $[e, \gamma]$ is within Q of a point in $C_\Gamma(g)$. We build the following geodesic quadrilateral:



There exists K such that $d(\sigma_\gamma(i), g\sigma_\gamma(i)) \le K|g|$. Let $h = \sigma_\gamma(i)^{-1}g\sigma_\gamma(i)$. Then h and g are conjugate, so there exists $x \in \Gamma$ with $hx = xg$, and $|x| \le |g| + |h| + c_0 \le |g| + K|g| + c_0 =: Q$.

We claim that $\sigma_\gamma(i)x \in C_\Gamma(g)$. We have

$$\begin{aligned} \sigma_\gamma(i)xg &= \sigma_\gamma(i)hx \\ &= \sigma_\gamma(i)\sigma_\gamma(i)^{-1}g\sigma_\gamma(i)x \\ &= g\sigma_\gamma(i)x. \end{aligned}$$

This completes the proof. □

17. November 4, 2010

Proposition. *If N is a non-trivial, finitely generated, normal subgroup of a finitely generated free group F, then $[F : N] < \infty$.*

A generalization is Greenberg's lemma:

Proposition. *If Γ is hyperbolic and H is an infinite, quasi-convex subgroup, then $[N_\Gamma(H) : H] < \infty$. In particular, if H is also normal, then $[G : H] < \infty$.*

Theorem. *If H is an infinite subgroup of a hyperbolic group Γ, then H contains an element of infinite order.*

Open Question. *Do all hyperbolic groups have a finite index, torsion-free subgroup?*

Open Question. *Can a* CAT(0) *group contain an infinite torsion subgroup?*

Definition. Suppose (Γ, A) is a finitely generated group. We say that Γ has the *quasi-monotone conjugacy property* (we wiill abbreviate this as QMC) if there exists a constant $K = K(A)$ such that, whenever $u, v \in F(A)$ with $\varepsilon(u)$ conjugate to $\varepsilon(v)$ in Γ, there exists a word $w = a_1 \cdots a_n$ with $w^{-1}uw = v$ in Γ and

$$d(e_\Gamma, w_i^{-1}uw_i) \leq K \max\{|u|, |v|\},$$

where $w_i = a_1 \cdots a_i$ for $i = 1, \ldots, n$.

Proposition. *Suppose (Γ, A) is a finitely generated group. If Γ has solvable word problem and the QMC property, then Γ has solvable conjugacy problem.*

Proof. For each $n \in \mathbb{Z}^+$, consider $B(n) = \{w \in F(A) \mid \ell(w) \leq n\}$. Since Γ has a solvable word problem, we can take $v_1, v_2 \in B(n)$ and decide if there exists an element $a \in A^{\pm 1}$ such that $a^{-1}v_1a = v_2$ in Γ. If there does exist such an a, then we'll say $v_1 \sim v_2$.

We construct graphs $G(n)$ with vertex set $V = B(n)$ and edges determined by the relation $\sim$. The QMC property tells us that $u, v \in F(A)$ are conjugate in Γ if and only if u, v lie in the same path component of $G(n)$ for $n = K \max\{|u|, |v|\}$.

The graph $G(n)$ has at most $(2|A|)^n$ vertices. Any injective edge-path of length ℓ has length at most $(2|A|)^n$. An edge path of length ℓ between u and v gives a conjugating word. So $u, v \in F(A)$ are conjugate in Γ if and only if there exists a conjugating word w with $|w|_A \leq (2|A|)^{K \max\{|u|,|v|\}}$. (We can also write this bound as $\mu^{\max\{|u|,|v|\}}$ where $\mu = (2|A|)^K$.) ☐

Remark. Recall: we already proved that for Γ hyperbolic, we can find a much better bound on the length of the conjugating word. In fact, we showed there is a global constant c_0 such that we can find a word conjugating u to v of length at most $|u| + |v| + c_0$. But the above proves the conjugacy problem for a larger class of groups.

Definition. $w \in \{A \cup \overline{A}\}^*$ is *cyclically reduced* if $a_i \neq a_{i+1}^{-1}$ and $a_1 \neq a_n^{-1}$.

For example, if $w = bab^{-1}babb^{-1}$ can reduce to baa or aab, and these are conjugate. Any word cyclically reduces to a cyclically reduced word that is unique up to cyclic permutation of the letters.

Let F_n be a free group. How do we solve the conjugacy problem? Take $u, v \in F_n$, and replace them with cyclically reduced words. Then check if the resulting words are cyclic permutations of each other.

Exercise. Figure out the running time. (Hint: linear?)

In a hyperbolic group, replace the notion of cyclically reduced words with words whose cyclic permutations are all $(8\delta + 1)$-local geodesics. We will call words satisfying this property *geometrically cyclically reduced*. It turns out that it takes no more than $|u| + |v|$ steps to replace u and v with geometrically cyclically reduced words. (This comes from the Dehn algorithm.)

Lemma. *There exists a finite set of words Σ such that if u and v are geometrically cyclically reduced, then u and v are conjugate if and only if they are conjugate by a sequence of words from Σ where the length of the sequence is bounded by* $\max\{|u|, |v|\}$.

18. November 9, 2010

Definition. Let Γ be a finitely generated group with finite generating set A. Let $u \in F(A)$. Then the *cone of* u is

$$\mathrm{Cone}(u) = \{v \in F(A) \mid uv \text{ is geodesic in } \Gamma\}.$$

Observe that if u is not geodesic in Γ, then $\mathrm{Cone}(u) = \emptyset$. If u is geodesic in Γ, then $\mathrm{Cone}(u) \neq \emptyset$ since it contains at least the empty string. Observe also that $\mathrm{Cone}(u)$ only depends on $\varepsilon(u)$.

If $\gamma \in \Gamma$, then we define the *cone of* γ as $\mathrm{Cone}(\gamma) = \mathrm{Cone}(u)$ where u is any geodesic representative of γ.

Example. Let $\Gamma = F_2$, $A = \{a, b\}$. Let $\gamma = ba$. Then $\mathrm{Cone}(\gamma)$ is the set of strings that do not begin with a^{-1}. In fact, there are exactly five different cone types:

$$\mathrm{Cone}(\gamma) = \{\text{all strings that do not begin with the inverse of the last letter of } \gamma\}.$$

Note that in the case $\gamma = e$, $\mathrm{Cone}(\gamma) = F(A)$. In general, there are $2n + 1$ distinct cone types in F_n. In fact, there are $2n + 1$ cone types for $\mathbb{Z}^n$ if we use a basis as our generating set.

Definition. Given a constant $k > 0$, the *k-tail* is the set

$$k\text{-tail}(\gamma) = \{h \in \Gamma \mid d(e, \gamma h) < d(e, \gamma), d(e, h) \leq k\}.$$

Remark. Observe that there are only finitely many possible k-tails.

Theorem. *If Γ is hyperbolic, then Γ has finitely many cone types (for any Cayley graph).*

Proof. Given $\gamma \in \Gamma$, the idea is that to determine $\mathrm{Cone}(\gamma)$, we need to know which strings to avoid locally at γ.

Let $k = 2\delta + 2$. Then the k-tail of γ determines $\mathrm{Cone}(\gamma)$. That is, if γ, γ' have the same k-tail, then they have the same cone type.

Suppose $\gamma, \gamma' \in \Gamma$ with $k\text{-tail}(\gamma) = k\text{-tail}(\gamma')$. Let $v \in \mathrm{Cone}(\gamma)$. If $|v| = 0$, then $\overline{v} = e$, and $v \in \mathrm{Cone}(\gamma')$. If $|v| = 1$, then $v \in \mathrm{Cone}(\gamma')$ by construction. (Follow the definitions.)

We will now induct on the length of v. We claim that if $v \in \mathrm{Cone}(\gamma)$ and $v \in \mathrm{Cone}(\gamma')$ and $a \in A^{\pm 1}$ with $va \in \mathrm{Cone}(\gamma)$, then $va \in \mathrm{Cone}(\gamma')$. Suppose $va \notin \mathrm{Cone}(\gamma')$. Then there exists a word $w \in F(A)$ with $|w| < d(e, \gamma') + |v'| + 1$ and $\varepsilon(w) = \varepsilon(\gamma' va)$. Write $w = w_1 w_2$ where $|w_1| = d(1, \gamma) - 1$, and then $|w_2| \leq |v| + 1$. Now the edge paths $u'v$ and $w_1 w_2$ are geodesics that end 1 unit apart. By a lemma from last week, $u'v$ and w_1w_2 are $(2\delta+1)$-uniformly close. Hence $d(\overline{w}_1, \gamma') < 2\delta+2$. Notice that $(\gamma')^{-1}\overline{w}_1 \in k$-tail$(\gamma')$. So $(\gamma')^{-1}\overline{w}_1) \in k$-tail$(\gamma)$ by assumption.

Let p be a geodesic path to $\gamma(\gamma')^{-1}\overline{w}_1$. Since $(\gamma')^{-1}\overline{w}_1 \in k$-tail$(\gamma)$, $d(\gamma(\gamma')^{-1}\overline{w}_1, e) < d(e,\gamma)$. Now notice that the path pw_2 is a path from e to γva:

$$\gamma(\gamma')^{-1}\overline{w}_1\overline{w}_2 = \gamma(\gamma')^{-1}\gamma'\overline{v}a = \gamma\overline{v}a.$$

Moreover, the length of pw_2 is shorter than $d(e,\gamma) + |v| + 1$. This contradicts the fact that $u\overline{v}a$ is geodesic. Hence Γ has finitely many cone types. $\square$

Proposition. *If Γ is an infinite group with finitely many cone types, then Γ contains an element of infinite order.*

Proof. Since Γ is infinite, there exists a geodesic edge path in a Cayley graph of Γ that begins at e and has length greater than the number of cone types. Let w label such a path. Break up $w = u_1u_2u_3$, where the shared end of u_1 and u_2 is γ_1 and the shared end of u_2 and u_3 is γ_2, so that $\mathrm{Cone}(\gamma_1) = \mathrm{Cone}(\gamma_2)$. (These two vertices must exist by the pidgeonhole principle.)

Since w is geodesic, $u_2u_3 \in \mathrm{Cone}(\gamma_1) = \mathrm{Cone}(\gamma_2)$. Then $u_2^2u_3 \in \mathrm{Cone}(\gamma_1)$, so $u_1u_2^2u_3$ is geodesic. Iterating, for all $n > 0$ we have $u_1u_2^nu_3$ is geodesic. Since subwords of geodesics are geodesics, $\overline{u}_2$ is an infinite order element of Γ, and we are done. $\square$

Proposition. *If Γ is hyperbolic and $H \leq \Gamma$ is infinite and quasi-convex, then $[N_\Gamma(H) : H] < \infty$. In particular, if $H \triangleleft \Gamma$, then H is finite index.*

Remark. In the case of free groups, this is called Greenberg's lemma.

To prove the above, we will need the following ingredient:

Proposition. *If Γ is hyperbolic, then for all $R > 0$, there exist only finitely many conjugacy classes $[\gamma]$ with* translation length $\tau_\Gamma(\gamma) \leq R$.

19. November 16, 2010

Definition. Let (Γ, A) be a finitely generated group. For $\gamma \in \Gamma$, define the *translation number of γ* as follows:

$$\tau_{\Gamma,A}(\gamma) = \lim_{n\to\infty} \frac{d(e_\Gamma, \gamma^n)}{n} = \lim_{n\to\infty} \frac{|\gamma^n|}{n}.$$

Remark. This limit always exists, because word length is subadditive. It is a classical fact that if $f\colon \mathbb{N} \to \mathbb{N}$ is subadditive (that is, $f(m+n) \leq f(m) + f(n)$), then $\lim_{n\to\infty} f(n)/n$ exists.

Question. *We know that Γ acts on $\Delta(\Gamma, A)$ geometrically. Define*

$$T(\gamma) = \inf\{d(x, \gamma x) \mid x \in \Delta\}.$$

Is $T(\gamma) = \tau(\gamma)$?

Proposition. *Some properties of the translation number:*

1. *$\tau_\Gamma(\gamma)$ only depends on the conjugacy class of γ.*
2. *$\tau_\Gamma(\gamma^m) = |m| \cdot \tau_\Gamma(\gamma)$.*
3. *If $H < \Gamma$ is finitely generated and QI embedded, then there exists K such that*
$$\frac{1}{K}\tau_\Gamma(h) \leq \tau_H(h) \leq K\tau_\Gamma(h).$$

Proposition. *If Γ is hyperbolic with finite generating set A, then for all $R > 0$ there exist only finitely many conjugacy classes $[\gamma]$ such that $\tau_{\Gamma,A}(\gamma) \leq R$.*

Proof. Let $\gamma \in \Gamma$, and let u be a minimal length word in $[\gamma]$. If $|u| > 8\delta + 1 := k$, then the path β determined by the powers u^n is a quasi-geodesic. Recall that k-local geodesic implies quasi-geodesic for $k > 8\delta + 1$.)

Observe that $n|u| = d_\beta(e, u^n)$. We have

$$\frac{1}{\lambda}|u^n| - \lambda \leq n|u| \leq \lambda|u^n|$$

(Recall from our proof that k-local geodesics are quasi-geodesics, we do not need the extra λ on the right.) Now $|u|/\lambda \leq |u^n|/n$, so

$$\tau_\Gamma(u) = \lim_{n\to\infty} \frac{|u^n|}{n} \geq \frac{|u|}{\lambda}.$$

Since there are only finitely many words of length at most a fixed R, this completes the proof. □

Theorem (Delzant)**.** *If (Γ, A) is a hyperbolic group, then $\{\tau_{\Gamma,A}(\gamma) \mid \gamma \in \Gamma\}$ has the following property. There exists $N \in \mathbb{N}$ such that for all $\gamma \in \Gamma$, $N\tau_{\Gamma,A}(\gamma) \in \mathbb{N}$.*

Proposition. *Let Γ be a hyperbolic group and $H \leq \Gamma$ infinite and quasi-convex. Then $[N_\Gamma(H) : H] < \infty$. In particular, if $H \triangleleft \Gamma$, then $[\Gamma : H] < \infty$. (This is a generalization of Greenberg's lemma for free groups.)*

Proof. We will show there exists a constant $D \geq 0$ such that $d(\gamma_0, H) \leq D$ for all $\gamma_0 \in N_\Gamma(H)$.

Since H is quasi-convex, H is finitely generated, hyperbolic, and QI embedded. Since H is an infinite order hyperbolic group, there exists an $\alpha \in H$ of infinite order. There exists a constant k_1 such that if $g \in C_\Gamma(\alpha)$, then $d(g, \alpha) < k_1$ (since $[C_\Gamma(\alpha), \langle\alpha\rangle] < \infty$). And there exists a constant k_2 such that if $\gamma^{-1}\alpha\gamma \in H$, then $\tau_H(\gamma^{-1}\alpha\gamma) \leq k_2\tau_\Gamma(\gamma^{-1}\alpha\gamma) = k_2\tau_\Gamma(\alpha)$. Also noticce that if $\gamma^{-1}\alpha\gamma \in H$, then there are only finitely many H-conjugacy classes with $\tau_H < k_2\tau_\Gamma(\alpha)$. Let $c_1^{-1}\alpha c_1, \ldots, c_n^{-1}\alpha c_n$ be the conjugacy classes.

Suppose $\gamma_0 \in N_\Gamma(H)$. Then $\gamma_0^{-1}\alpha\gamma_0 \in H$, so there exists $h \in H$ and i such that $\gamma_0^{-1}\alpha\gamma_0 = h^{-1}(c_i^{-1}\alpha c_i)h$. Now

$$\alpha = \gamma_0 h^{-1}(c_i^{-1}\alpha c_i)h\gamma_0^{-1},$$

so $\gamma_0 h^{-1}c_i^{-1} \in C_\Gamma(\alpha)$. It follows that $d(\gamma_0 h^{-1}c_i^{-1}, \langle\alpha\rangle) < k_1$.

Since $\gamma_0 \in N_\Gamma(H)$, we can write $\gamma_0 h^{-1} = h'\gamma_0$ for some h'. So

$$\begin{aligned}
d(\gamma_0, H) &= d(h'\gamma_0, H) \\
&\leq d(g'\gamma_0, h'\gamma_0 c_i^{-1}) + d(h'\gamma_0 c_i^{-1}, H) \\
&= |c_i| + d(\gamma_0 h^{-1}c_i^{-1}, H) \\
&\leq |c_i| + d(\gamma_0 h^{-1}c_i^{-1}, \langle\alpha\rangle) \\
&\leq \max\{|c_i|\} + k_1.
\end{aligned}$$

□

Theorem (Gromov, Delzant)**.** *If Γ is a hyperbolic group, then for any finite set $\{h_1, \ldots, h_r\} \subset \Gamma$ there exists $n \in \mathbb{N}$ such that $\langle h_1^n, \ldots, h_r^n\rangle$ is free of rank at most r.*

Theorem (E. Rips)**.** *Every hyperbolic group Γ acts on a simplicial complex P with the following properties:*

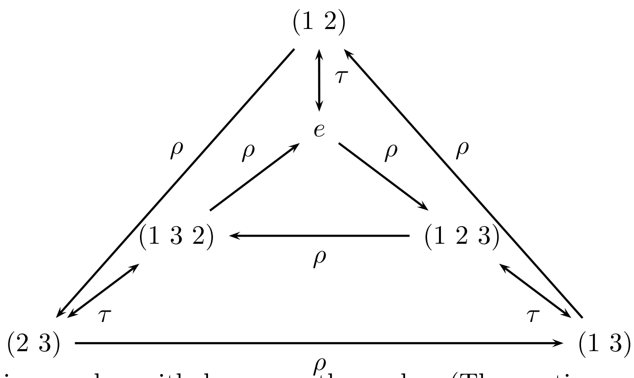
(1) *P is finite-dimensional, locally finite, and contractible.*
(2) *Γ acts simplicially with compact quotient* $\Gamma \setminus P$ *and finite stabilizers.*
(3) *Γ acts freely and transitively on the vertices.*
(∗) *If Γ is torsion free, then the quotient* $\Gamma \setminus P$ *is a* $K(\Gamma, 1)$.

Proof. We'll only describe the complex. Fix a generaing set A for Γ. For each $R > 0$, we construct a simplicial complex $P_R(\Gamma)$ as follows. $V(P_R) = \Gamma$ for all R. $\{x_0, \dots, x_n\} \subset \Gamma$ determines an n-simplex in P_R if $\operatorname{diam}_{\Gamma,A}\{x_0, \dots, x_n\} \leq R$. If R is large enough, then P_R is contractible. □

20. November 18, 2010

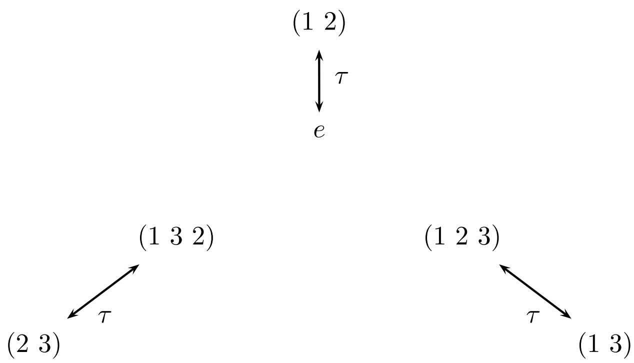
Let H be a finitely generated subgroup of Γ generated by a subset $S' \subset S$ of the generating set for Γ. When we erase the edges in $S \setminus S'$, then we're left with connected components in bijective correspondence with the cosets of H in G. The coset containing the identity is a Cayley graph for H.

Example. Consider $S_3 = \langle \rho, \tau \rangle$, and $N = \langle p \rangle$. The Cayley graph of S_3 is given by:



Then $N \setminus \Delta$ is an edge with loops on the ends. (The vertices are in bijective correspondence with the cosets of N in Γ.)

Example. Now let $H = \langle \tau \rangle$. Deleting the ρ edges:



The quotient $\Delta(S_3) \setminus H$ is just a triangle.

Proposition. *Suppose* $1 \to N \to \Gamma \to B \to 1$ *is an exact sequence of finitely generated infinte groups with* Γ *hyperbolic. Then* N *cannot be* ϵ*-quasi-convex.*

Proof. Suppose N is ϵ-quasi-convex. N acts on any $\Delta(\Gamma)$. Consider the quotient $\Lambda = N \setminus \Delta$. This is infinite and locally finite (it's a covering space action, so the vertices in the quotient will have the same valence as in the cover). Because B is infinite, we can find arbitrarily long edge paths in the quotient. Let $p\colon \Delta \to \Lambda$ be the quotient map, and let β be a path from $p(e)$ to some point b in Λ of length at least $k = 2\epsilon + 2\delta$. Choose $u \in N$ with $|u| > 2k + 2\delta$.

Consider a geodesic $\widehat{\beta}$ from e to some preimage $c := p^{-1}(b)$. Then $\{e, u, uc, c\}$ forms a geodesic quadrilateral. Let r be the midpoint of $[e, u]$ and s the midpoint of $[uc, c]$. Then $d(r, s) < 2\delta$. Since N is ϵ-QC, there is an n_1 within ϵ of r and n_2 within ϵ of s, with $n_1 \in N$, $n_2 \in cN$. Now $d(n_1, n_2) < 2\delta + 2\epsilon$. Since $p(n_1) = p(e)$ and $p(n_2) = b$, we have found a shorter path than β from $p(e)$ to b, contradicting our choice of β. □

We'll now define the boundary of a proper, geodesic, δ-hyperbolic metric space. Here are the properties we want:

(1) $\partial(F_n) \cong \text{Ends}(F_n)$
(2) $\partial(\mathbb{H}^2) \cong S^1$
(3) $\partial(\mathbb{E}^2) \cong S^1$
(4) $\overline{X} := X \cup \partial(X)$ should be compact, with X open and dense in $\overline{X}$
(5) $\pi\colon \partial(X) \twoheadrightarrow \text{Ends}(X)$, where the point preimages should be connected components
(6) any isometry of X extends to a homeomorphism of $\partial(X)$
(7) the homeomorphism type of $\partial(X)$ is a QI invariant (true for hyperbolic spaces, but false for CAT(0) spaces)

Let X be a geodesic metric space with fixed base point $x_0 \in X$. We might define the boundary of X as the set of geodesic rays in X based at x_0 with a topology so that two rays are close if they stay close for a long time. It is clear that $\partial(F_n) \cong \text{Ends}(F_n)$ with this definition.

Definition. Let X be a proper, geodesic, δ-hyperbolic metric space. Suppose $\gamma_1, \gamma_2\colon [0, \infty) \to X$ are geodesic rays. We say that they are *asymptotic*, denoted $\gamma_1 \sim \gamma_2$, if there exists $k > 0$ such that for all $t \geq 0$, $d(\gamma_1(t), \gamma_2(t)) \leq k$. (Equivalently, $\gamma_1 \sim \gamma_2$ if their images in X have finite Hausdorff distance.)

Definition. Let X be a proper, geodesic, δ-hyperbolic metric space. Let $w \in X$ be a fixed base point. The *boundary of* X is $\partial_w(X) = \{[\gamma] \mid \gamma(0) = w\}$, where $[\gamma]$ is the asymptotic class of geodesic rays.

We write $\partial(X) = \{[\gamma] \mid \gamma \text{ geodesic}\}$, where γ is a geodesic ray with unspecified base point. It can be shown that $\partial_w(X) = \partial(X)$, so this is an equivalent definition of the boundary.

Definition. In any metrix space X with fixed base point $w \in X$, we can define the *Gromov inner product*:

$$(x \cdot y)_w = \frac{1}{2}[d(x, w) + d(y, w) - d(x, y)].$$

If we draw the triangle w, y, x and collapse to a tripod, the Gromov inner product is the distance from w to the center vertex.

If X is a geodesic, δ-hyperbolic metric space, then for all x, y, z:

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (z \cdot y)_x\} - \delta.$$

In a δ-hyperbolic metric space, $(x \cdot y)_w$ measures how long it takes $[w, x]$ and $[w, y]$ to get more than δ apart.

Definition. Let $\{x_i\} \subset X$ be a sequence. We say that it *converges to* ∞, denoted $x_i \to \infty$, if $\liminf_{i,j}(x_i \cdot x_j)_w = \infty$. Two sequences $\{a_i\}$ and $\{b_i\}$ are *related* if $\liminf_i(a_i \cdot b_i)_w = \infty$. We write $\{a_i\}R\{b_i\}$. This is in fact an equivalence relation. The *boundary of* X is

$$\partial(X) = \{[\{x_n\}] \mid x_n \to \infty\}.$$

21. November 23, 2010

Throughout today, X is a proper, geodesic, δ-hyperbolic metric space. Recall from last time:

$$\begin{aligned}
\partial^g X &= \{[\gamma] \mid \gamma\colon [0, \infty) \to X \text{ geodesic}\} \\
\partial^g_p X &= \{[\gamma] \mid \gamma\colon [0, \infty) \to X \text{ geodesic}, \gamma(0) = p\} \\
\partial^q X &= \{[r] \mid r\colon [0, \infty) \to X \text{ quasi-geodesic}\}
\end{aligned}$$

Recall the notation $\gamma(\infty) := [\gamma]$.

Lemma. *The natural map $f\colon \partial^g X \to \partial^q X$ is a bijection. For each $p \in X$ and $z \in \partial^g X$ there exists a geodesic ray $c\colon [0, \infty) \to X$ with $c(0) = p$ and $c(\infty) = z$.*

Proof. f is trivially an injection (since any geodesic ray is a quasi-geodesic ray). Now let $p \in X$ and $r\colon [0, \infty) \to X$ a quasi-geodesic ray. Let $c_n = [p, r(n)]$. Since X is proper we can use the Arzela–Ascoli theorem to get a subsequence of (c_n) that converges to a ray $c\colon [0, \infty) \to X$. We have $c(0) = p$. Moreover, there exists K such that for all n, the image of c_n is K-close to the image of r. So $\operatorname{im}(c)$ is also K-close to $\operatorname{im}(r)$, hence $c(\infty) = r(\infty)$. □

In fact, this shows that all three of the point sets above are equal. So we will write only ∂X for the boundary from now on. We will describe a topology on the boundary to satisfy the following. Fix a basepoint $p \in X$ and let c_1 and c_2 be geodesic rays based at p. Then c_1 and c_2 are "close" if they stay (δ-)close for a long time.

Lemma (Reverse triangle inequality)**.** *If a, b, c are the lengths of the sides of a geodesic triangle, then $|b - c| \le a$.*

Lemma. *Suppose $c, c'\colon [0, T] \to X$ are geodesics with $c(0) = c'(0)$. If there exists $K > 0$ and $t_0 \in [0, t]$ such that $d(c(t_0), \operatorname{im}(c')) \le K$, then $d(c(t), c'(t)) \le 2\delta$ for all $t \le t_0 - K - \delta$.*

Proof. Let $t \in [0, T]$ such that $d(c(t_0), c'(t)) \le K$. For any $a < t_0 - K - \delta$, $c(a)$ is not δ-close to $[c'(t), c(t_0)]$. But for any such a there exists a' such that $d(c(a), c'(a)) \le \delta$. So $|a - a'| \le \delta$ by the reverse triangle inequality. So $d(c(a), c'(a)) \le 2\delta$. □

Lemma. *If $c_1, c_2\colon [0, \infty) \to X$ are geodesic rays with $c_1(0) = c_2(0)$ and $c_1(\infty) = c_2(\infty)$, then $d(c_1(t), c_2(t)) \le 2\delta$ for all $t \ge 0$.*

Proof. Choose K such that $\operatorname{im}(c_1)$ and $\operatorname{im}(c_2)$ are K-Hausdorff close. Given t, choose $t_0 > t + K + \delta$. Since $c_1(\infty) = c_2(\infty)$ □

Lemma. *If $c_1, c_2\colon [0, \infty) \to X$ are geodesic rays with $c_1(\infty) = c_2(\infty)$, then there exist T_1, T_2 such that $d(c_1(T_1 + t), c_2(T_2 + t)) \le 5\delta$ for all $t \ge 0$.*

Sketch of proof. Using Arzela–Ascoli, build c_1' with $c_1'(0) = c_1(0)$ and $c_1'(\infty) = c_1(\infty)$ as the limit of sequences of geodesics from $c_1(0)$ to $c_2(n)$. □

Lemma. *If $z \neq w \in \partial X$, then there exists a geodesic line $c\colon \mathbb{R} \to X$ with $c(\infty) = z$ and $c(-\infty) = w$.*

Proof. We'll prove this in detail later. Brief sketch: fix a point p. Pick a geodesic $c_1 = [p, w]$ and another geodesic $c_2 = [p, z]$. Go out along each geodesic far enough to be more than δ apart, and connect c_1 and c_2 be a geodesic from $c_1(t)$ to $c_2(t)$. Any geodesic from w to z will have to go through a δ-ball around $c_1(t)$. Use Arzela–Ascoli twice to show a sequence of geodesics converges to a geodesic line. □

We'll topologize $\overline{X} = X \cup \partial X$ as follows.

Definition. A *generalized ray* is a geodesic $c\colon I \to X$ with either $I = [0, \infty)$ or $I = [0, R]$. In the latter case, we extend c to $[R, \infty)$ by the constant map at $c(R)$.

Definition. $\overline{X} = \{c(\infty) \mid c \text{ is a generalized ray}\}$ (where $c(\infty)$ means the endpoint for a generalized ray of the form $c\colon [0, R] \to X$).

Definition. A sequence $\{a_i\} \subset X$ *converges to* ∞ if

$$\lim_{i,j\to\infty} (a_i \cdot a_j)_p = \infty.$$

We say that two sequences which converge to ∞ are *related*, denoted $\{a_i\}R\{b_i\}$, if

$$\lim_{i\to\infty} (a_i \cdot b_i) = \infty.$$

We write $\partial^s X = \{[\{a_i\}] \mid \{a_i\} \to \infty\}$.

Recall the Gromov inner product:

$$(a \cdot b)_p = \frac{1}{2}[d(a, p) + d(b, p) - d(a, b)].$$

This measure how long the geodesics $[p, a]$ and $[p, b]$ stay δ-close. It also measures $d(p, [a, b])$, since $d(p, [a, b]) \leq (a \cdot b)_p + \delta$. (Note that this last inequality explains why we care about sequences converging to ∞. If they don't, then any geodesic from a_i to b_i will have to go through some particular δ-ball at distance $\sup(a \cdot b)_p$ away from p. If the sequences do converge to ∞, then $\lim d(p, a_i) = \infty$.)

Example. The relation R needs δ-hyperbolicity to be transitive. Consider the Cayley graph $\Delta(\mathbb{Z} \oplus \mathbb{Z}, S = \{x, y\})$, with $a_n = x^n$, $b_n = y^n$, and $c_n = x^n y^n$. Let $w = (0, 0)$. We have $\{a_i\} \to \infty$, $\{b_i\} \to \infty$, and $\{c_i\} \to \infty$. It is easy to calculate

$$(a_i \cdot b_i)_w = 0$$
$$(a_i \cdot c_i)_w = i$$
$$(b_i \cdot c_i)_w = i$$

so $\{a_i\}R\{c_i\}$, $\{b_i\}R\{c_i\}$, but $\{a_i\}\not R\{b_i\}$.

Recall that in a δ-hyperbolic space, for all x, y, z,

$$(x \cdot y)_z \geq \min\{(x \cdot z)_w, (z \cdot y)_w\} - \delta.$$

We define a basis for a topology on $\overline{X}$ as follows. If $x \in X$, use the $B(x, \epsilon)$. If $x \in \partial X$, define

$$(x \cdot y)_p = \inf_{\substack{x_i \to x \\ y_i \to y}} \{\liminf_i (x_i \cdot y_i)_p\},$$

and let $N(x,k) = \{y \in \overline{X} \mid (\cdot y) > k\}$ for $x \in \partial X$.

Example. To see why we need both the liminf and the inf, consider the Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}_2$.

22. November 30, 2010

Recall $\overline{X} = X \cup \partial(X)$. We wrote $\partial^s X = \{[\{x_n\}] \mid x_n \to \infty\}$, where $x_n \to \infty$ means $(x_i \cdot x_j)_w \to \infty$ for some base point w as $i, j \to \infty$.

We will extend the Gromov inner product to ∂X as follows. Let $x, y \in \overline{X}$. Let $w \in X$ be a fixed base point. Define:

$$(x \cdot y)_s := \inf_{\substack{x_i \to x \\ y_i \to y}} \{\liminf_i (x_i \cdot y_i)_w\}.$$

Proposition. *If $x, y \in X$, then $(x \cdot y)_s = (x \cdot y)_w = (x \cdot y)$, where we drop the base point in the right hand side since the basepoint will not matter.*

Proof. Suppose $x_i \to x$ and $y_i \to y$ in X. We have

$$\begin{aligned}
|(x \cdot y) - (x_i \cdot y)| &= \frac{1}{2}|d(x,w) + d(y,w) - d(x,y) - d(x_i,w) - d(y,w) + d(x_i,y)| \\
&\leq \frac{1}{2}[|d(x,w) - d(x_i,w)| + |d(x_i,y) - d(x,y)|] \\
&\leq \frac{1}{2}[d(x,x_i) + d(x,x_i)] \\
&= d(x,x_i).
\end{aligned}$$

Now

$$\begin{aligned}
|(x \cdot y) - (x_i \cdot y_i)| &\leq |(x \cdot y) - (x_i \cdot y)| + |(x_i \cdot y) - (x_i \cdot y_i)| \\
&\leq d(x,x_i) + d(y,y_i) \\
&\to 0.
\end{aligned}$$

□

Remark. $\{x_n\} \to \infty$ is equivalent to $d(w,[x_i,x_j]) \to \infty$. If we have two sequences $\{x_n\}$ and $\{y_n\}$ converging to different points on the boundary, then we can draw geodesics $[x_i, y_i]$ and take the limit (using AA arguments) to get an ideal triangle $\{w, x, y\}$, where $x_n \to x \in \partial X$ and $y_n \to y \in \partial X$. Then $|(x \cdot y)_s - d(w,[x,y])| < 2\delta$. So the extended Gromov inner product still measures distance from the third side of the triangle.

We define a basis for a topology on $\overline{X}$ as follows.

(1) For $x \in X$, $r > 0$, take $B(x,r) = \{y \in X \mid d(x,y) < r\}$.
(2) For $x \in \partial X$, $k > 0$, take $N_{x,k} = \{y \in \overline{X} \mid (x \cdot y)_s > k\}$.

Let $\mathcal{B}$ be the collection of open sets given above.

Remark. Consider a geodesic ray c, and let $c(\infty) = x \in \partial X$. Consider the 2δ ball around $c(n)$. Define a set

$$V_n(c(\infty)) = \{c'(\infty) \mid c' \colon I \to X \text{ geodesic}, c'(0) = w, d(c'(n), c(n)) < 2\delta\}.$$

These are not quite a basis for the point x. But every V_n contains some $N_{x,k}$, and the sets V_n can be more useful in proofs.

Proposition. *The following are properties of the extended Gromov inner product on $\overline{X}$. Let $x, y \in \overline{X}$.*

(1) *$(x \cdot y)_s = \infty$ if and only if $x, y \in \partial X$ and $x = y$.*
(2) *If $x \in \partial X$ and $\{x_i\} \subset X$, then $(x_i \cdot x_s \to \infty$ if and only if x_i converges to ∞ and $[\{x_i\}] = x$.*
(3) *If $x, y \in \overline{X}$, then there exist sequences $\{\overline{x}_i\} \to x$, $\{\overline{y}_i\} \to y$ such that $(x \cdot y)_s = \lim_{i\to\infty}(\overline{x}_i \cdot \overline{y}_i)$. If $x, y \in X$, then the constant sequences work.*
(4) *If $x, y \in \partial X$ and $x_i \to x$, $y_i \to y$, then*

$$(x \cdot y)_s \leq \liminf_i (x_i \cdot yy_i) \leq (x \cdot y)_s + 2\delta.$$

(5) *Let $x, y \in \overline{X}$ and $y_i \to y$. Then $\liminf_i (x \cdot y_i)_s \geq (x \cdot y)_s$.*
(6) *For $x, y, z \in \overline{X}$,*

$$(x \cdot y)_s \geq \min\{(x \cdot z)_s, (y \cdot z)_s\} - \delta.$$

Proof. Will be sent out. □

Proposition. *$\mathcal{B}$ forms a basis for a topology on $\overline{X}$.*

Proof. It is trivial that $\mathcal{B}$ covers $\overline{X}$. Let $B_1, B_2 \in \mathcal{B}$. Then we must show there exists $B_3 \in \mathcal{B}$ with $B_3 \subset B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then this is trivial. If B_1 is of type (1) and B_2 is of type (2), then $B_1 = B(x, \epsilon)$ and $B_2 = N(z, k)$. Let $y \in B_1 \cap B_2$. Since $y \in B_1$, $y \in X$. So there exists ϵ_1 such that $B(y, \epsilon_1) \subset B(x, \epsilon) = B_1$. Since $y \in B_2$, we know $(y \cdot z) > k$. So there exists ϵ_2 such that $(y \cdot z) > k + \epsilon_2 > k$. Let $\epsilon' = \min\{\epsilon_1, \epsilon_2\}$.

Claim. *$B(y, \epsilon') \subset B_1 \cap B_2$.*

Proof. It is clear that $B(y, \epsilon') \subset B_1$. Choose $p \in B(y, \epsilon')$. Choose a sequence $z_i \to z$ such that $\lim_{i\to\infty}(p \cdot z_i) = (p \cdot z)$. (We can choose such a sequence by property 3.) Then

$$|(z_i \cdot p) - (z_i \cdot y)| \leq d(p, y) < \epsilon' \leq \epsilon_2.$$

Then

$$\begin{aligned} -\epsilon_2 &\leq \liminf_i (z_i \cdot p) - \liminf_i (z_i \cdot y) \leq \epsilon_2 \\ \Rightarrow -\epsilon_2 &\leq (z \cdot p) - (z \cdot y) \end{aligned}$$

(Think about this.) It will follow that

$$(z \cdot p) = (z \cdot p) - (z \cdot y) + (z \cdot y) \geq -\epsilon_2 + (z \cdot y) > -\epsilon_2 + k + \epsilon_2 = k. \qquad \square$$

The case where B_1 and B_2 are both of type (2) will be sent out. □

To prove $\overline{X}$ is compact, we will do the following.

(1) Prove $\overline{X}$ is metrizable: we will show separable and first countable (hence second countable) and regular (hence metrizable by Urysohn metrization theorem). Proving regularity requires case by case checking.
(2) Prove $\overline{X}$ is sequentially compact. (Hint: AA.)

Remark. There is a map $\partial X \to \mathrm{Ends}(X)$ which is continuous and surjective, and the point preimages are the connected components of $\partial(X)$. In particular, we can detect when a group is 1-ended by checking whether the boundary is connected.

23. December 2, 2010

Theorem. *If X, X' are proper, geodesic, δ-hyperbolic metric spaces and $f\colon X \to X'$ is a QI-embedding, then there exists a map $\partial f\colon \partial X \to \partial X'$ that is an (topological) embedding. If f is a QI, then ∂f is a homeomorphism.*

Proof. Pick base points $p \in X$ and $p' = f(p) \in X'$. Let $c_1, c_2\colon [0,\infty) \to X$ be geodesic rays with $c_1(0) = c_2(0) = p$ and $c_1(\infty) = c_2(\infty)$. Then $f \circ c_1$ and $f \circ c_2$ are quasi-geodesics in X'.

Pick geodesic rays $c_1' \in [f \circ c_1]$ and $c_2' \in [f \circ c_2]$ (that is, $(f \circ c_i)(\infty) = c_i'(\infty)$). Then $c_1(\infty) = c_2(\infty)$ if and only if $c_1'(\infty) = c_2'(\infty)$.

We define a map $\partial f\colon \partial X \to \partial X'$ by $\partial f(c) = c'$, where c' is a geodesic ray asymptotic to $f \circ c$. The above shows we have a well-defined injective function.

We now show continuity of the map. With c_i and c_i' as above, we will show that if $c_1 \in V_n(c_2)$ (that is, $c_1(n) \in B(c_2(n), 2\delta)$), then $c_1' \in V_{n'}(c_2')$. Since $f \circ c_1$ is a quasi-geodesic, there is some constant K and some t_0 such that $d(c_1'(t_0), f \circ c_1(n)) \le K$. We now have

$$d(c_1'(t_0), \operatorname{im} c_2') \le 2K + \lambda \cdot 2\delta + \lambda =: L.$$

It follows that $d(c_1'(t), c_2'(t)) \le 2\delta$ for all $t \le t_0 - L - \delta$. (We proved a lemma that said this.)

Now since $d(p', f \circ c_1(n)) \ge \frac{n}{\lambda} - \lambda$, we have $t_0 > \frac{n}{\lambda} - \lambda - K$. Now for $n' > \frac{n}{\lambda} - \lambda - K - L - \delta$, $c_1' \in V_{n'}(c_2')$.

If f is a QI, then there is a quasi-inverse f^{-1}, and the above argument shows continuity of the map ∂f^{-1}. This shows that ∂f is a homeomorphism. □

Let G be a word hyperbolic group. Then ∂G makes sense, and we may as well work in a Cayley graph. If $f\colon G \to G$ is a QI, then $\partial f\colon \partial G \to \partial G$ is a homeomorphism. In particular, G acts by isometries on any Cayley graph, so any $g \in G$ acts as a homeomorphism on ∂G.

Let $g \in G$ be an element of infinite order. We have already seen that $\{g^n \mid n \in \mathbb{Z}\}$ determines a quasi-line. That is, there is a QI embedding $\mathbb{Z} \to G$, hence there is an injection of $\partial\mathbb{Z}$, a discrete pair of points, into ∂G. Picking $e \in G$ as our base point, we can pick a geodesic line that is asymptotic to $\{g^n\}$, and this line goes to two distinct points in ∂G. Call them

$$g^+ = \lim_{i\to\infty} g^i$$
$$g^- = \lim_{i\to\infty} g^{-i}.$$

Since g leaves $\{g^n\}$ invariant, ∂g (the induced homeomorphism on the boundary) will fix $\{g^\pm\}$.

Remark. No parabolics.

Now let $z \in \partial G$, $z \ne g^\pm$.

Claim. *If $z \ne g^-$, then*

$$\lim_{i\to\infty} (\partial g)^i \cdot z = g^+.$$

Sketch of proof. $z \ne g^-$ implies that $\langle g^-, z\rangle < \infty$, and $\langle g^-, \partial g \cdot z\rangle < \langle g^-, z\rangle$. (We will use angled brackets for the Gromov inner product for now, since we used dot for the action.) Similarly, $\langle (\partial g)^i \cdot z, g^+\rangle \to \infty$ as $i \to \infty$. □

Theorem. *For all U, V open with $g^+ \in U$ and $g^- \in V$ with $U \cap V = \emptyset$, there exists N such that for all $n \geq N$:*

$$(\partial g)^n(\partial G - V) \subset U.$$

Proof. Given the above claim, this is just compactness. □

Theorem. *If G is hyperbolic with $|\partial G| > 2$, then G contains F_2.*

Sketch of proof. Ingredients:

(1) G is not infinite torsion, so there exists $g \in G$ with infinite order.
(2) G is not 2-ended (since $|\partial G| > 2$) and "no parabolics", so there exists an infinite order $h \in G$ with $\{h^\pm\} \cap \{g^\pm\} = \emptyset$.
(3) Use previous theorem to get sets appropriate for ping pong.

□

Theorem. *A much stronger version of the above theorem (same conditions as before): given a set $\{g_1, \ldots, g_r\}$ there exists an N such that $\{g_1^N, \ldots, g_r^N\} \cong F_{r'}$ for some $r' \leq r$.*

Definition. A word hyperbolic group G is *non-elementary* if G contains an F_2.

Remark. If G is a word hyperbolic group, then ∂G has 0, 2, or infinitely many points. If it has 0, it is finite. If it is nonempty, then G contains an infinite order element, hence ∂G has at least two points. If it has more than 2 points, then it contains an F_2, which has infinitely many points in its boundary.

Definition. If H is a subgroup of a word hyperbolic group G, then the *limit set of H* is

$$\Lambda(H) = \partial G \cap \overline{H},$$

where the closure of H is taken in $G \cup \partial G$.

Theorem. *The set $\{g^\pm \mid g \in G, o(g) = \infty\}$ is dense in ∂G.*

Proof. A geodesic ray in $\Delta(G, S)$ is some infinite geodesic word in $\{S \cup \overline{S}\}^*$. Let z be the boundary point the ray approaches, and pick a sequence of open sets (indexed by i) around it. We claim there is some g_i^+ in each open set, and $\lim g_i^+ = z$.

Take some beginning chunk of the infinite word z. There exists some M such that if a_M is the first M letters of this chunk and b_i is the rest, then $a_M(b_i)^r$ is a geodesic string for all r. (This is the ultra pumping lemma.) We will denote $\lim_{r\to\infty} a_M(b_i)^r := (a_M b_i a_M^{-1})^+$. Then take $g_i = a_M b_i a_M^{-1}$. □

Corollary. *If H is an infinite normal subgroup of a word hyperbolic group G, then $\Lambda(H) = \partial G$.*

Proof. Use $\partial g \cdot h^+ = (ghg^{-1})^+$. (Details left to the reader.) □