Math 250 - Homework 1

(1) Take the universal mapping property definition of a free group to do the following exercise: Suppose S is a subset of a group Γ such that $S \cap S^{-1} = \emptyset$. Then $\langle S \rangle$ is freely generated by S if and only if no reduced word $w = s_1 \cdots s_n$ in $\{S \cup \overline{S}\}^*$ is trivial in Γ. In particular, this shows the two definitions of free group that I gave are equivalent.

(2) Suppose $F_1 = F(S_1)$ and $F_2 = F(S_2)$ are free groups with bases S_1 and S_2 respectively. Show $F_1 \cong F_2$ if and only if $|S_1| = |S_2|$. If assuming the sets S_1 and S_2 makes you more comfortable, then you can assume that.

(3) Draw the Cayley graph of $\Gamma = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}_2$ (where the action of $\mathbb{Z}_2$ switches the factors) with generating set $S = \{a, b, c\}$ where a, b are the two standard generators of $\mathbb{Z} \oplus \mathbb{Z}$ and c generates the $\mathbb{Z}_2$ factor.

(4) Let F_2 be freely generated by $\{x, y\}$. Show that F_2 is also freely generated by $\{x, xy\}$.

(5) Show that F_2 has exactly three distinct subgroups of index 2.